Part IV:

Minimal Partitions and the Mumford-Shah Problem

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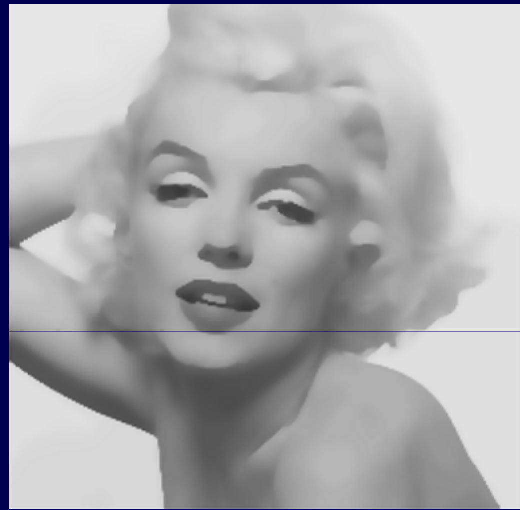
DAGM Tutorial „Convex Optimization“, Frankfurt 2011

Tutorial slides will be made available at:

http://cvpr.in.tum.de/tutorials/dagm2011

Overview

Convex multilabel optimization

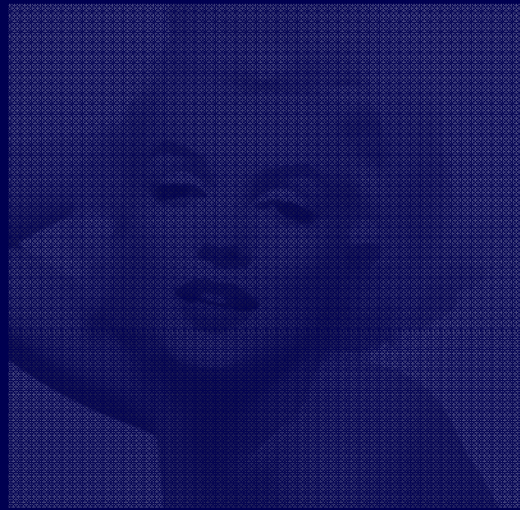


Piecewise smooth approximation

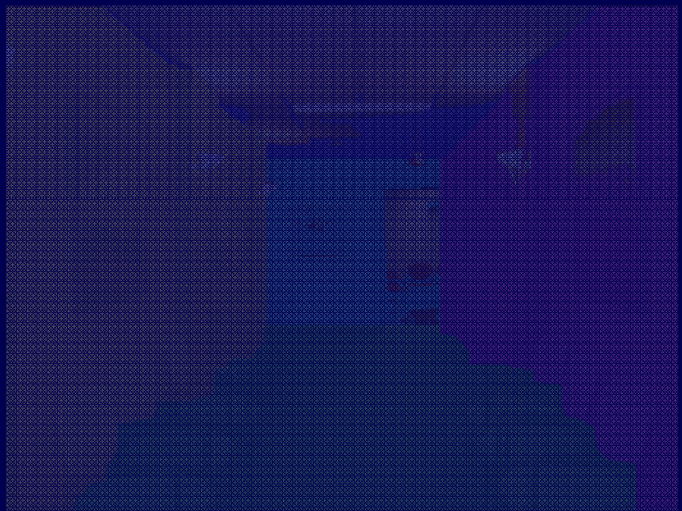
Convex ordering constraints

Convex optical flow

Convex multilabel optimization



Piecewise smooth approximation

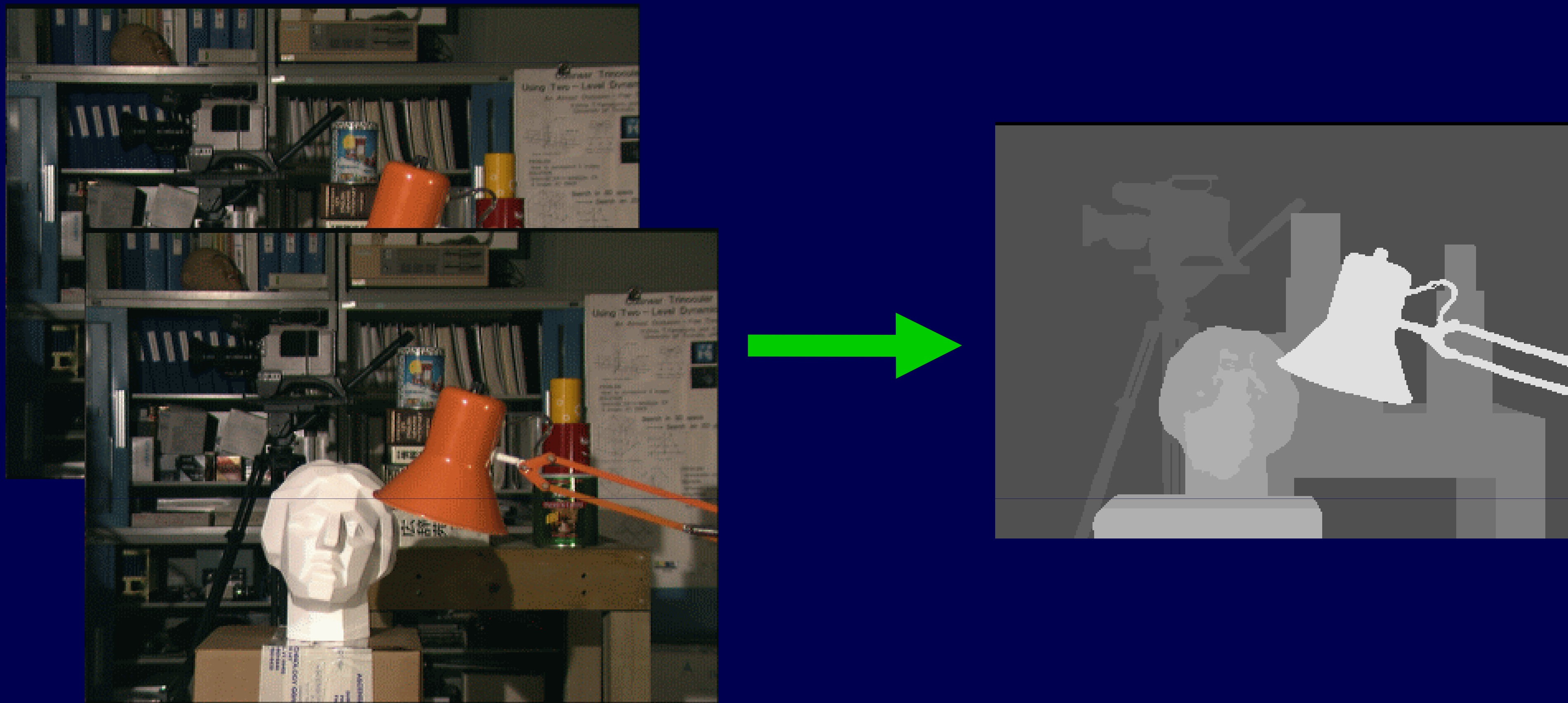


Convex ordering constraints

Convex optical flow

Multilabel Optimization

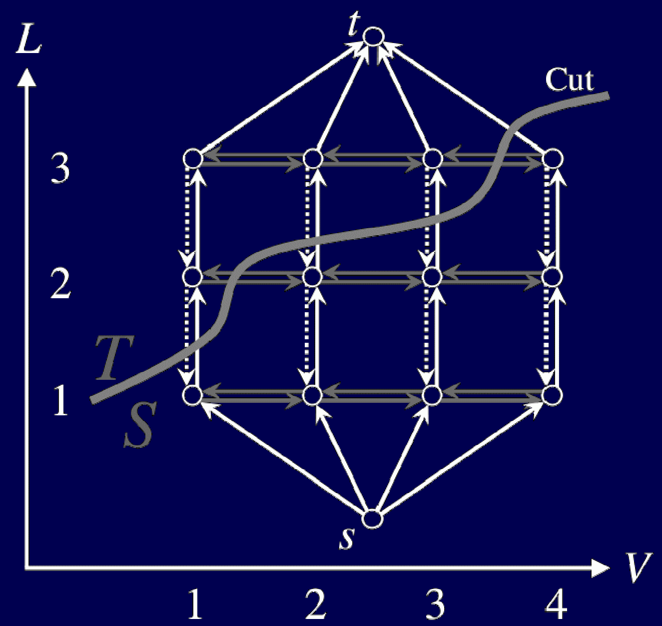
$$u : \Omega \rightarrow \Gamma = [\gamma_{min}, \gamma_{max}]$$



Example: Stereo

$$u : \Omega \rightarrow \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \underbrace{\int_\Omega \rho(u(x), x)\, dx}_{\text{data term}} + \underbrace{\int_\Omega |\nabla u(x)|\, dx}_{\text{label regularity}}$$



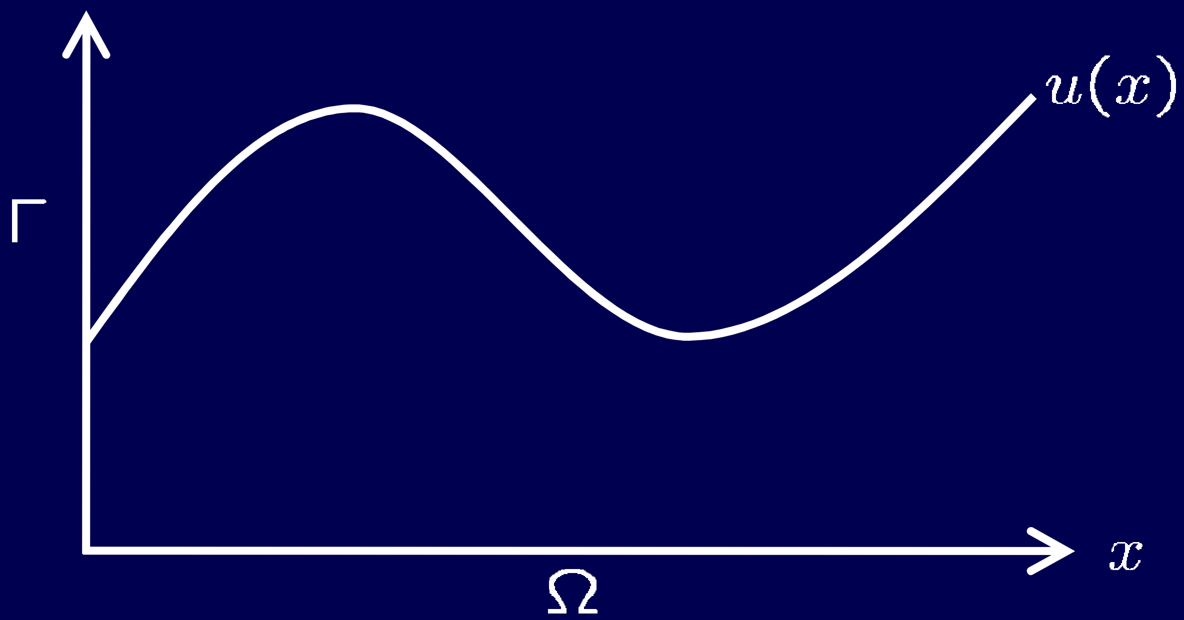
Spatially discrete solution: *Ishikawa 2003*

Drawbacks:

- requires lots of memory
- metrication errors
- no efficient parallel implem.

$$u : \Omega \to \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \underbrace{\int_\Omega \rho(x, u(x))\, dx}_{\text{nonconvex data term}} + \underbrace{\int_\Omega |\nabla u(x)|\, dx}_{\text{label regularity}} \qquad (*)$$

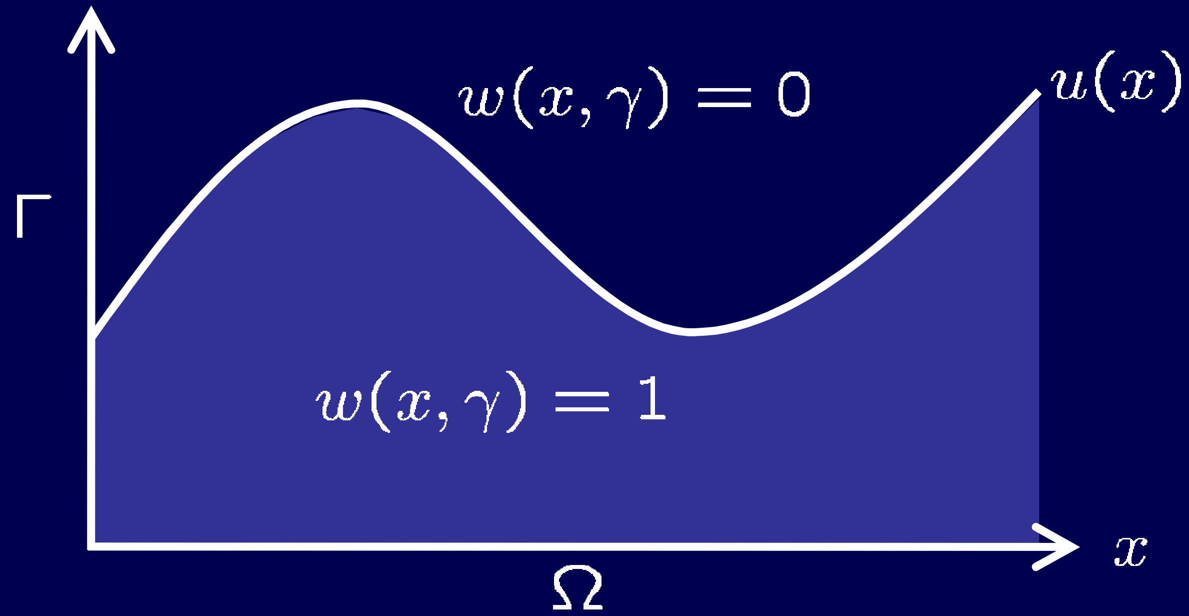


Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

$$u : \Omega \to \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \int_\Omega \rho(x, u(x))\, dx + \int_\Omega |\nabla u(x)|\, dx \qquad (*)$$

Let $w : (\Sigma = \Omega \times \Gamma) \to \{0, 1\}$ $\qquad w(x,\gamma) = 1_{u \geq \gamma}(x)$



Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

$$u : \Omega \to \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$\boxed{E(u) = \int_\Omega \rho(x, u(x))\, dx + \int_\Omega |\nabla u(x)|\, dx} \qquad (*)$$

nonconvex functional

Let $w : (\Sigma = \Omega \times \Gamma) \to \{0, 1\}$ $\qquad w(x, \gamma) = 1_{u \geq \gamma}(x)$

Theorem: Minimizing $(*)$ is equivalent to minimizing

$$\boxed{E(w) = \int_\Sigma \rho(x, \gamma)|\partial_\gamma w(x, \gamma)| + |\nabla w(x, \gamma)|\, dx\, d\gamma} \qquad (**)$$

convex functional

Solve $(**)$ in relaxed space $(w : \Sigma \to [0, 1])$ and threshold to obtain a globally optimal solution.

Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

Let

$$E(u) = \int_\Omega f(x, u, \nabla u)\, dx$$

be continuous in $x \in \mathbb{R}^d$ and u, and convex in ∇u.

Theorem:

For any function $u \in W^{1,1}(\Omega; \mathbb{R})$ we have:

$$E(u) = F(1_u) = \sup_{\phi \in \mathcal{K}} \int_{\Omega \times \mathbb{R}} \phi \cdot D\mathbf{1}_u,$$

where ϕ is constrained to the convex set

$$\mathcal{K} = \Big\{ \phi = (\phi^x, \phi^t) \in C_0\left(\Omega \times \mathbb{R}; \mathbb{R}^d \times \mathbb{R}\right) :$$
$$\phi^t(x,t) \geq f^*(x, t, \phi^x(x,t))\ ,\ \forall x, t \in \Omega \times \mathbb{R} \Big\}.$$

Pock, Cremers, Bischof, Chambolle, SIAM J. on Imaging Sciences '10

Optimality for Convex Regularizers

Therefore the functional $E(u)$ can be minimized by solving the relaxed saddle point problem

$$\min_v F(v) = \min_v \sup_{\phi \in \mathcal{K}} \int_{\Omega \times \mathbb{R}} \phi \cdot Dv,$$

Theorem:

The functional F fulfills a generalized coarea formula:

$$F(v) = \int_{-\infty}^{\infty} F(1_{v \geq s})\, ds.$$

As a consequence, we have a thresholding theorem assuring that we can globally minimize the functional $E(u)$.

Pock, Cremers, Bischof, Chambolle, SIAM J. on Imaging Sciences '10

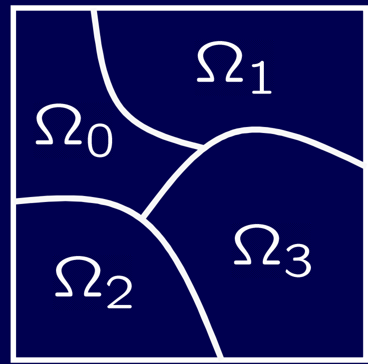
Evolution to Global Minimum

One of two input images

Courtesy of Microsoft Graz

Depth reconstruction

$$\min_{\Omega_0,\dots,\Omega_n} \frac{1}{2}\sum_i |\partial\Omega_i| + \sum_i \int_{\Omega_i} f_i(x)\,dx$$



$$\text{s.t.} \quad \bigcup_i \Omega_i = \Omega \subset \mathbb{R}^d, \quad \text{and } \Omega_i \cap \Omega_j = \emptyset \;\; \forall i \neq j$$

Potts '52, Mumford-Shah '89, Vese, Chan '02

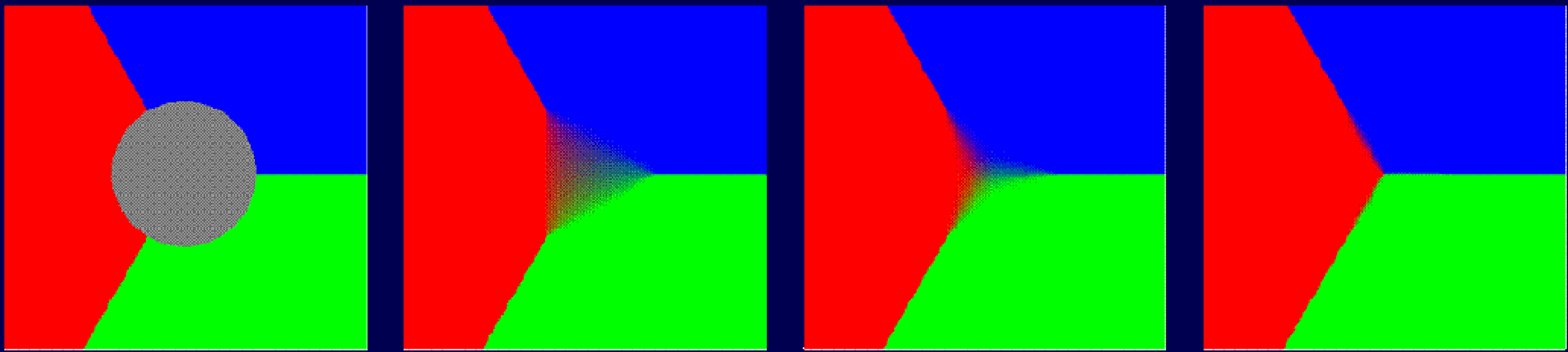
Proposition: With $v_i = 1_{\Omega_i}$, this is equivalent to

$$\min_{v\in\mathcal{B}} \frac{1}{2}\sum_i \int_\Omega |Dv_i| + \int_\Omega v_i\, f_i\, dx = \min_{v\in\mathcal{B}} \sup_{p\in\mathcal{K}} \sum_i \int_\Omega v_i \,\mathrm{div}\, p_i\, dx + \int_\Omega v_i\, f_i\, dx$$

where $\mathcal{K} = \left\{ p = (p_1,\dots,p_n)^\top \in \mathbb{R}^{n\times d} \;:\; \left|p_i - p_j\right| \leq 1,\; \forall i < j \right\}$

Chambolle, Cremers, Pock '08, Pock et al. CVPR '09

Test Case: The Triple Junction

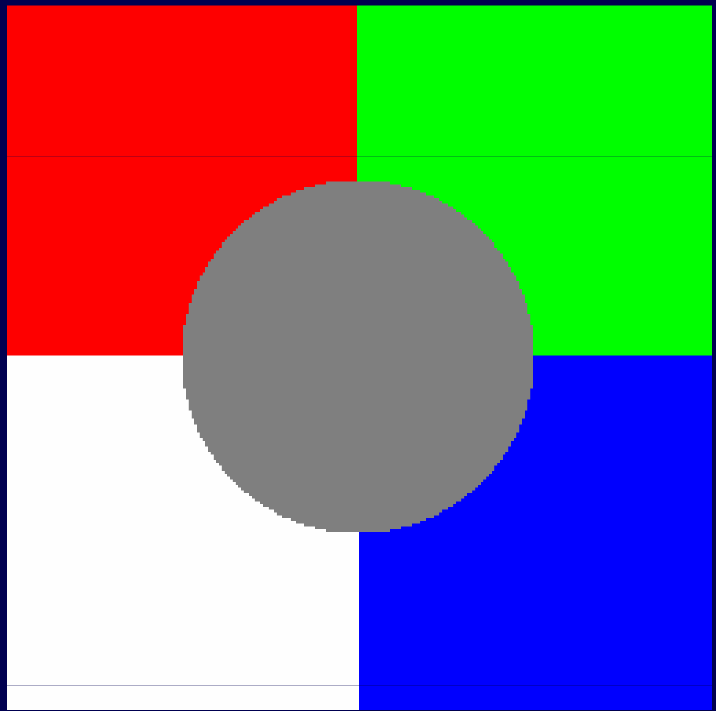


Input image Lellmann et al. '08 Zach et al. '08 our approach

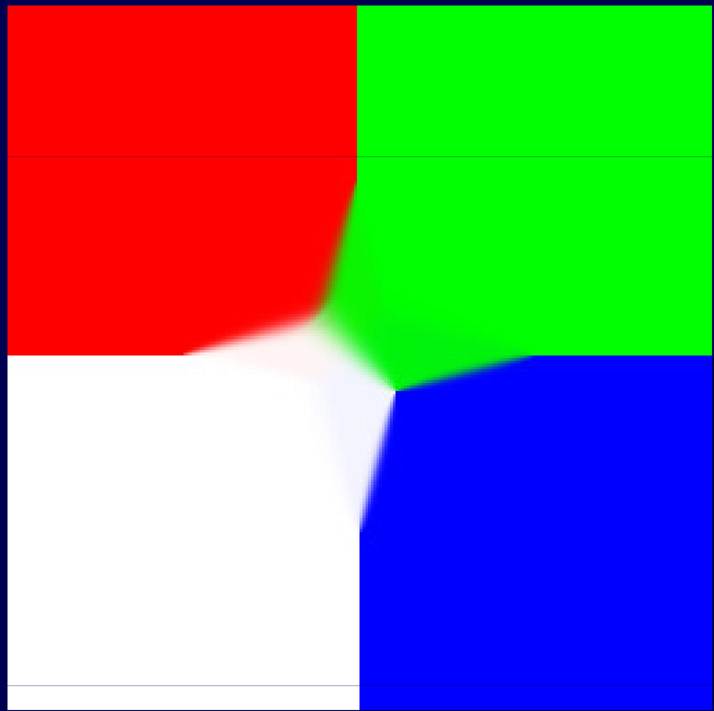
<u>Proposition:</u> The proposed relaxation strictly dominates existing relaxations.

Chambolle, Cremers, Pock '08, Pock et al. CVPR '09

Four-Region Case

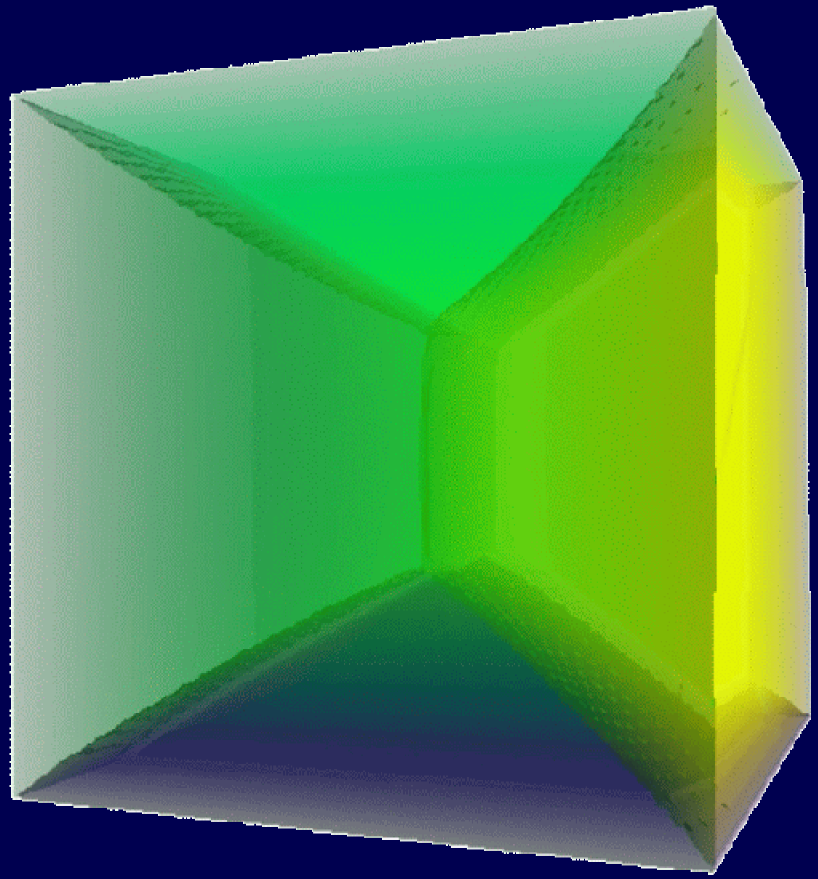


Input image

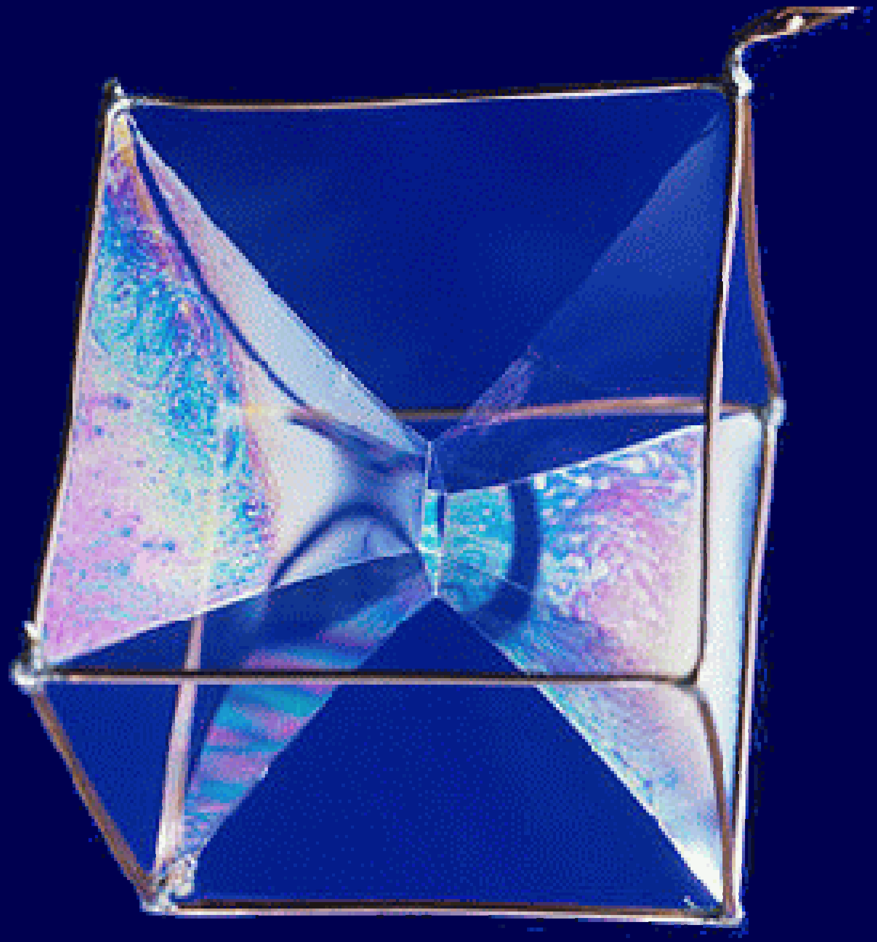


Inpainted

Chambolle, Cremers, Pock '08, Pock et al. CVPR '09



3D min partition inpainting



Soap film photo

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Multi-Region Segmentation

Input image

segmentation

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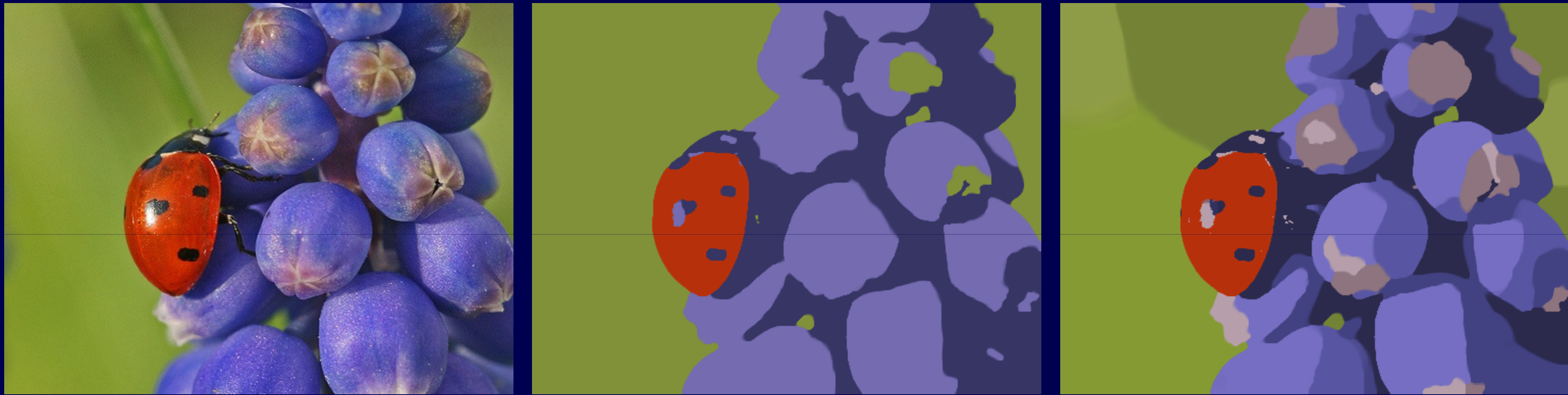
The Minimal Partition Problem



Input image | 5 label segmentation | 10 label segmentation

Chambolle, Cremers, Pock '08, Pock et al. CVPR '09

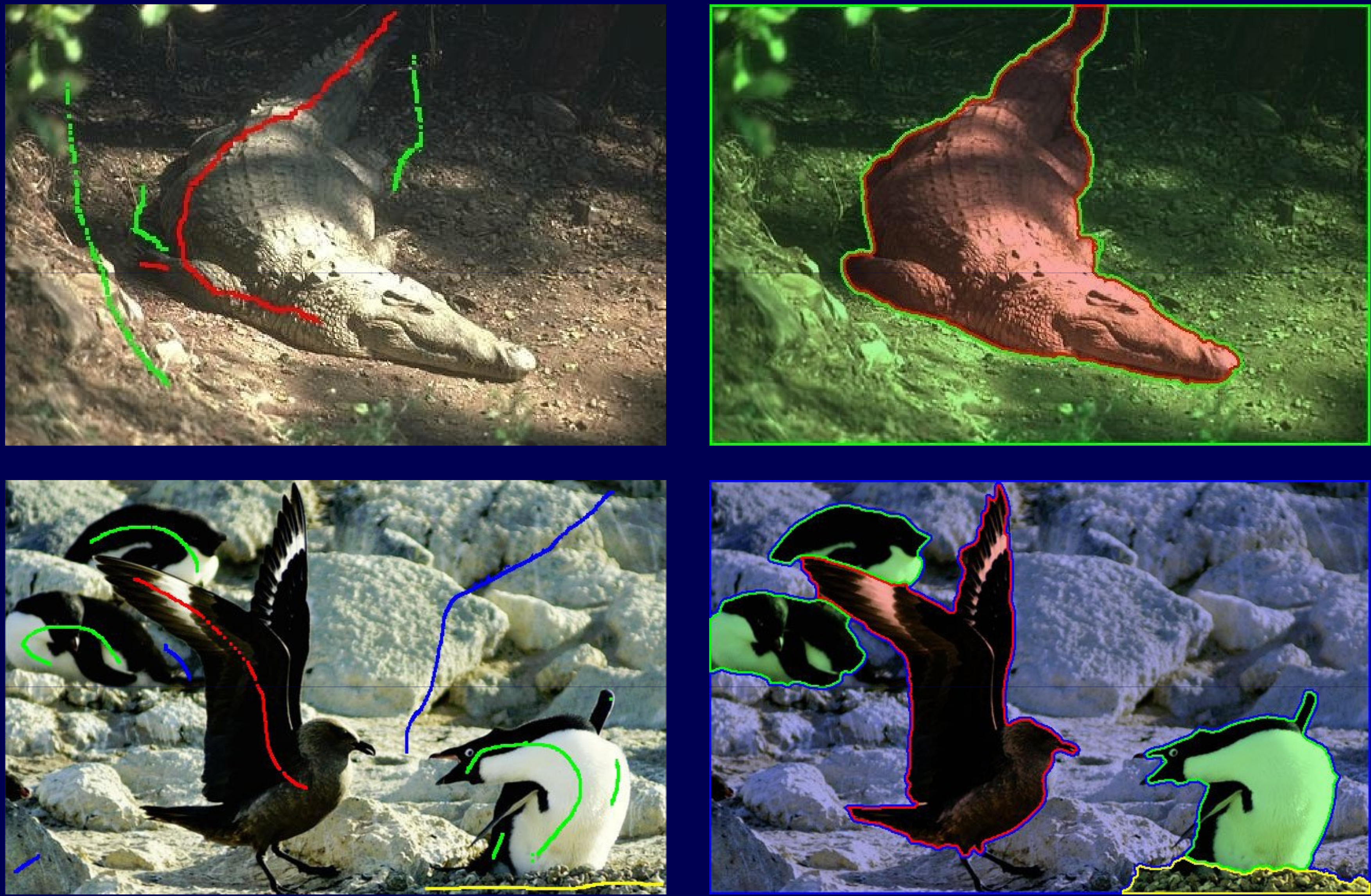
Multi-Region Segmentation

Input color image

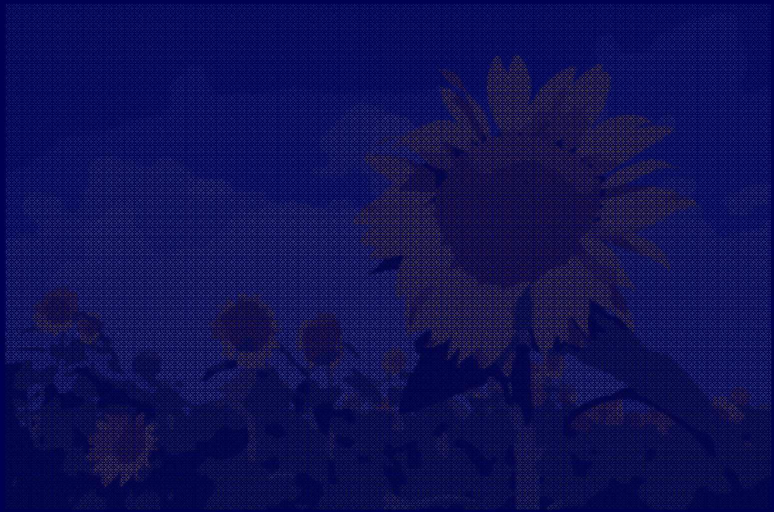
10 label segmentation

Chambolle, Cremers, Pock ’08, Pock et al. CVPR ’09

Interactive Segmentation

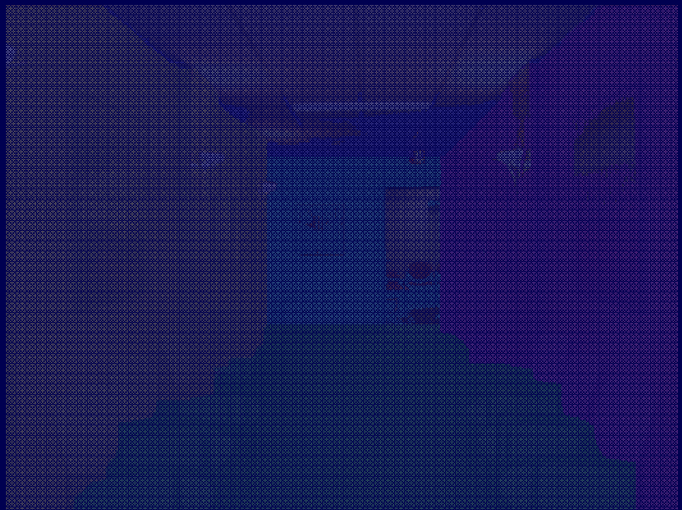


Nieuwenhuis, Toeppe, Cremers EMMCVPR '11



Convex multilabel optimization

Piecewise smooth approximation



Convex ordering constraints

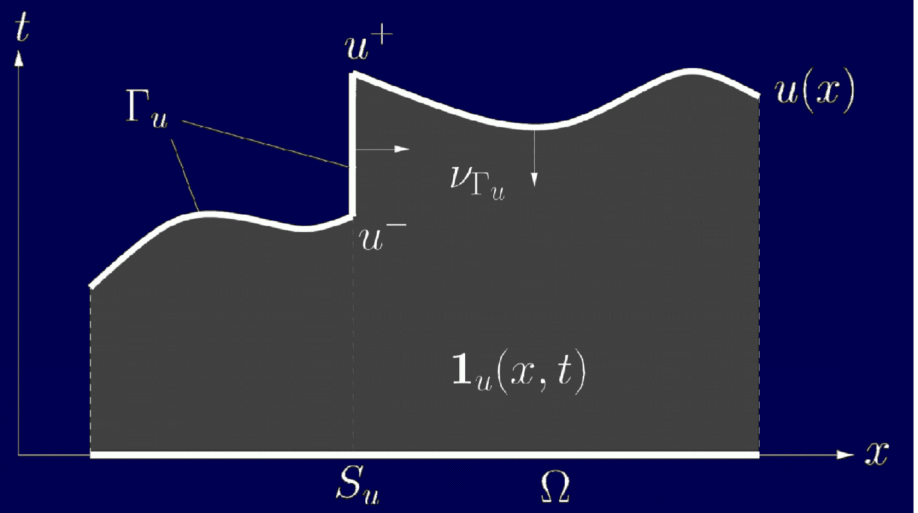
Convex optical flow

$$E(u) = \lambda \int_{\Omega} (f-u)^2 dx + \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \nu \mathcal{H}^1(S_u) \qquad (*)$$

Mumford, Shah '89

For $u \in SBV(\Omega)$, $(*)$ can be written as *(Alberti, Bouchitte, Dal Maso '04)*

$$E(u) = \sup_{\varphi \in K} \int_{\Omega \times \mathbb{R}} \varphi D\mathbf{1}_u,$$



with a convex set

$$K = \left\{ \varphi \in C_0(\Omega \times \mathbb{R}; \mathbb{R}^2) : \right.$$

$$\left. \varphi^t(x,t) \geq \frac{\varphi^x(x,t)^2}{4} - \lambda(t - f(x))^2, \quad \left| \int_{t_1}^{t_2} \varphi^x(x,s)ds \right| \leq \nu \right\},$$

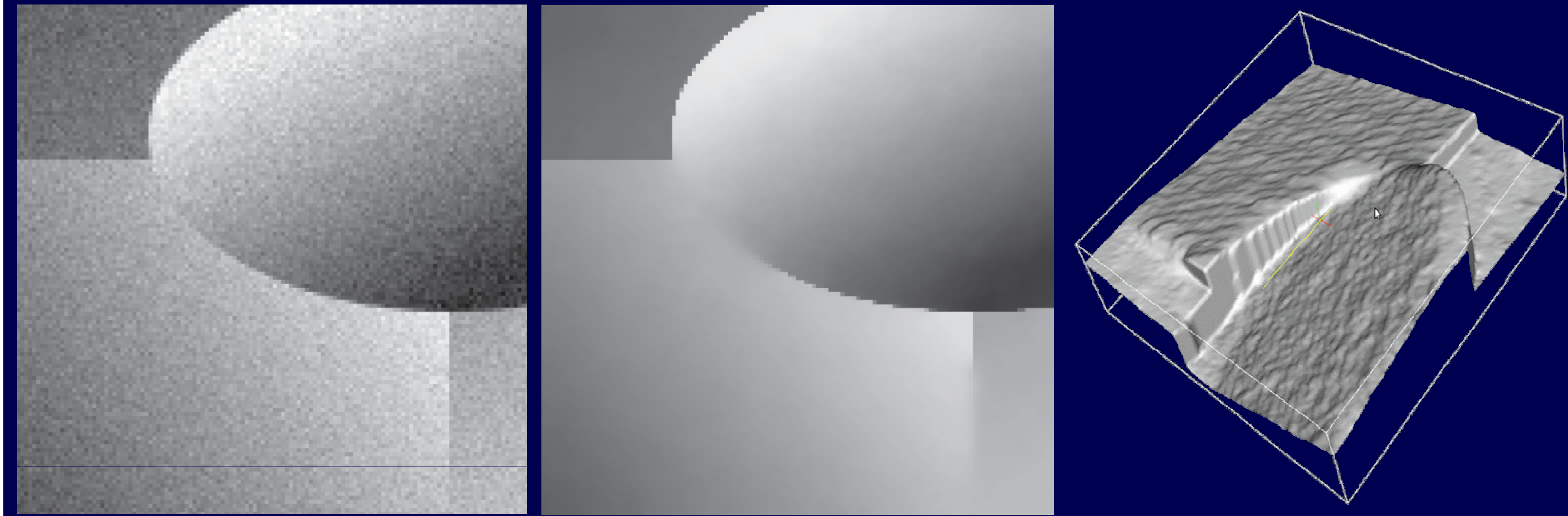
Pock, Cremers, Bischof, Chambolle ICCV '09

Input image

piecewise constant

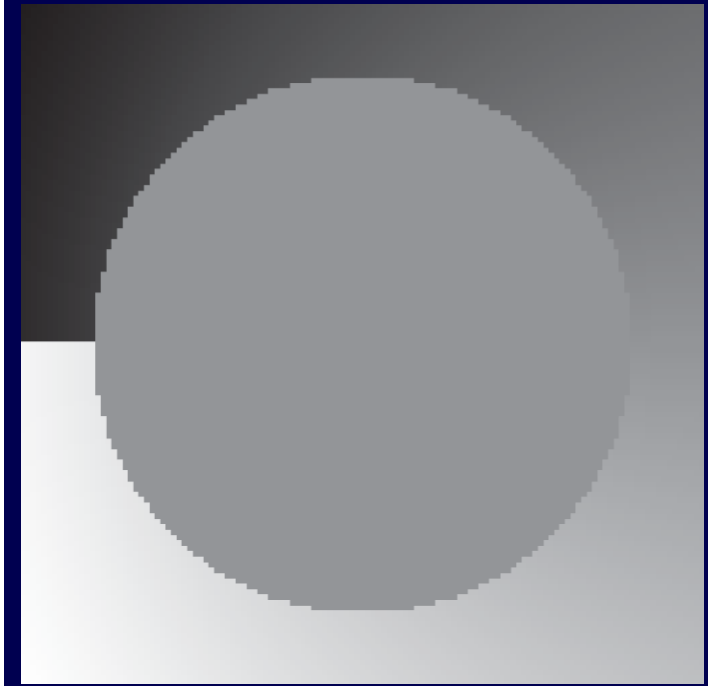
piecewise smooth

Pock, Cremers, Bischof, Chambolle ICCV '09

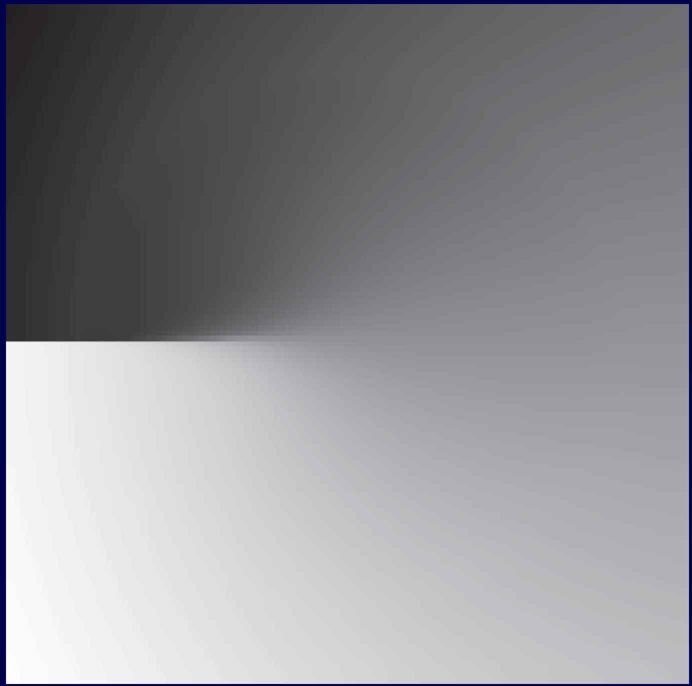


noisy input   restoration   surface plot

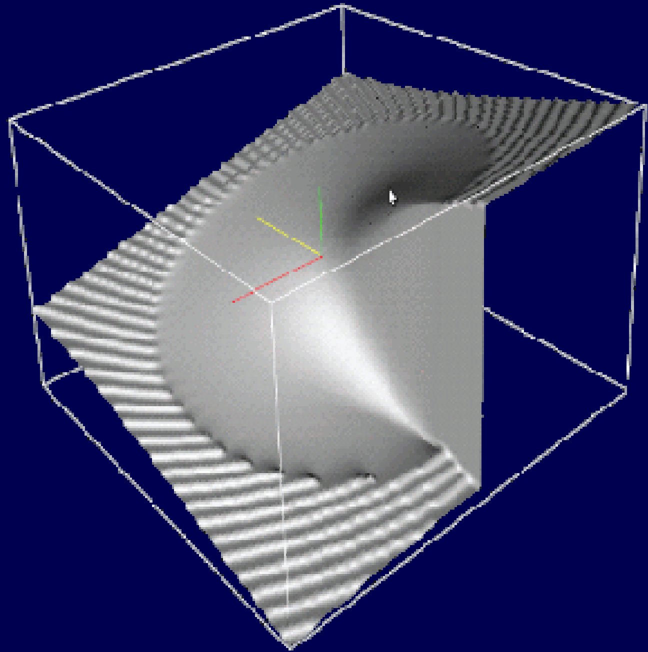
Pock, Cremers, Bischof, Chambolle ICCV '09



fixed boundary values



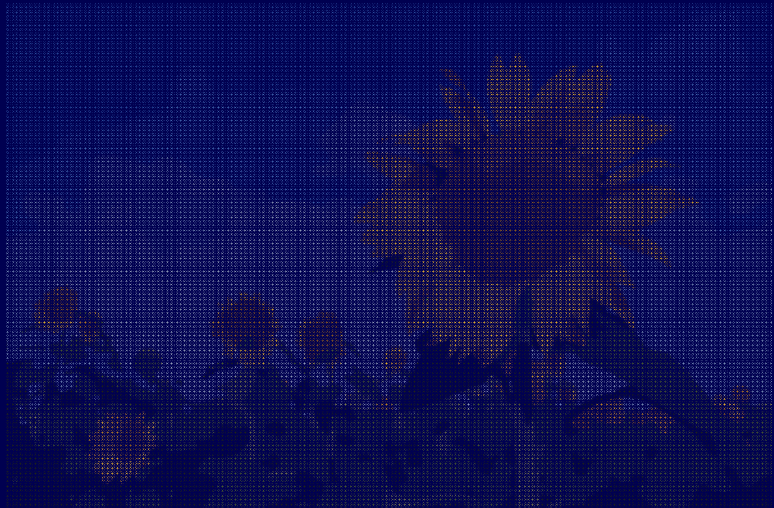
inpainted crack tip



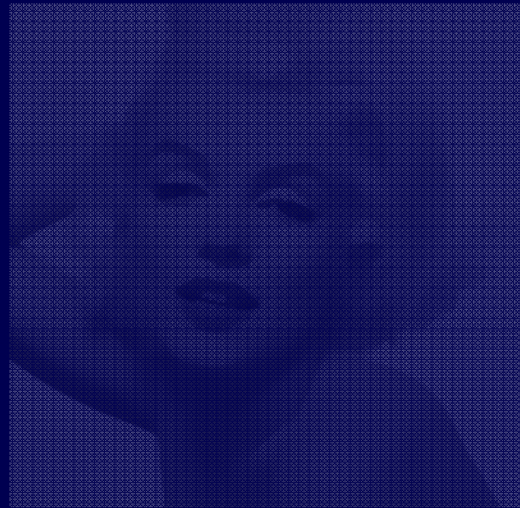
surface plot

Pock, Cremers, Bischof, Chambolle ICCV '09

Overview



Convex multilabel optimization



Piecewise smooth approximation

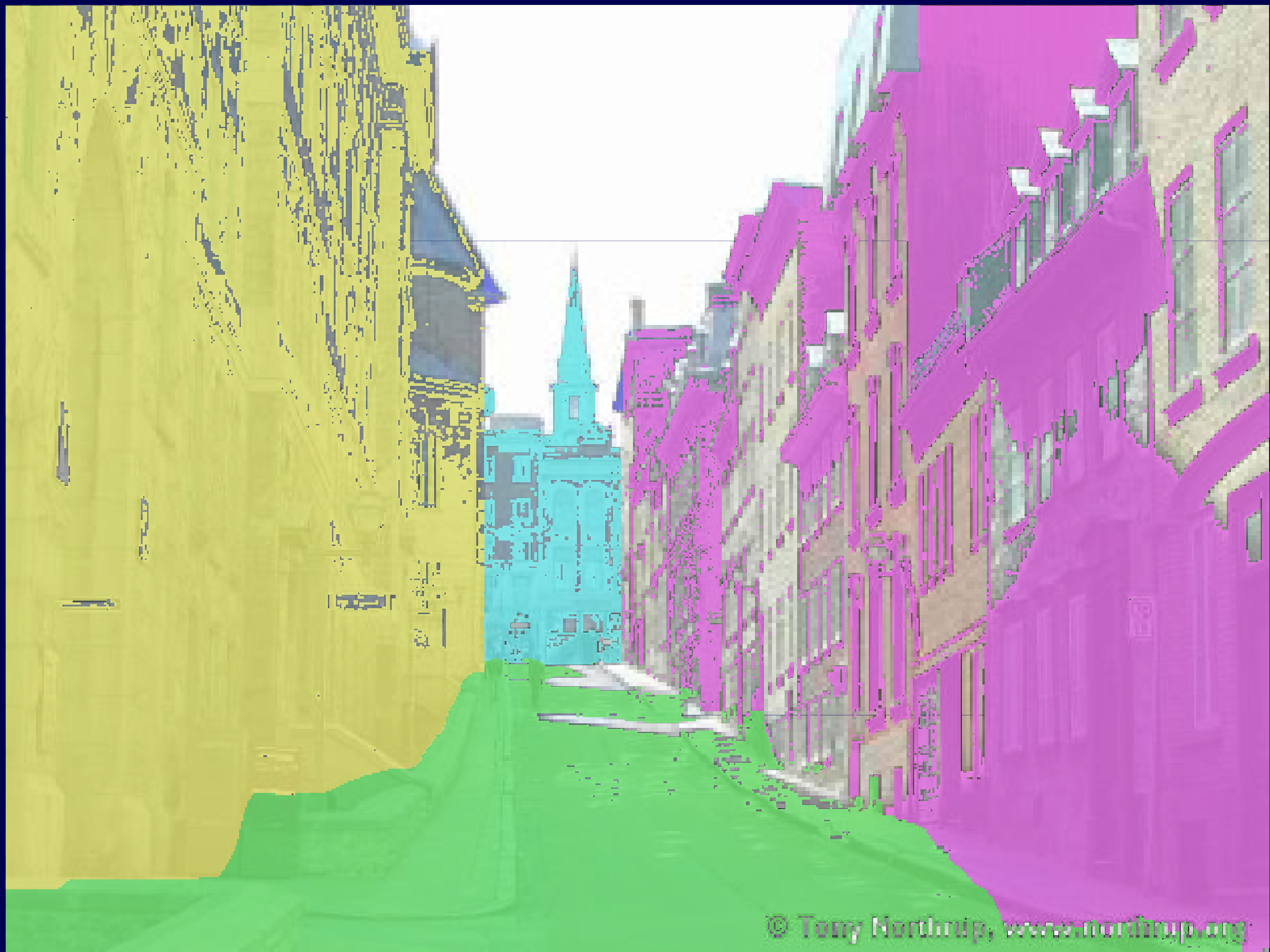
Convex ordering constraints

Convex optical flow

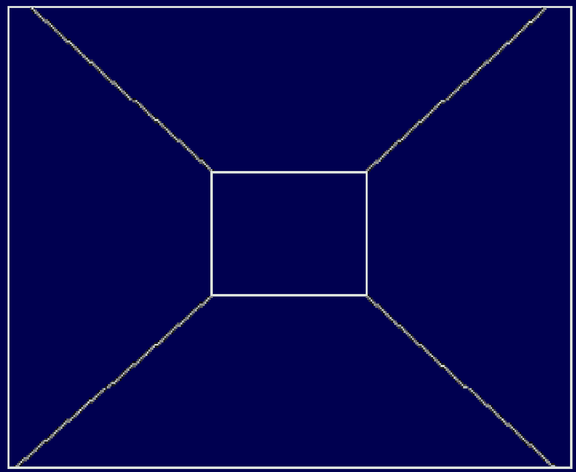
General Ordering Constraints

Strekalovskiy, Cremers, ICCV 2011

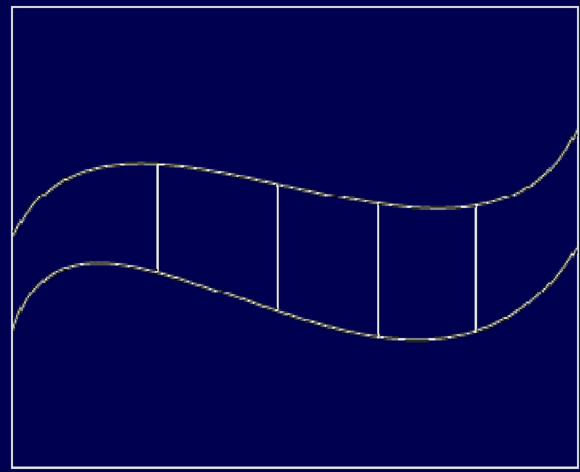
General Ordering Constraints



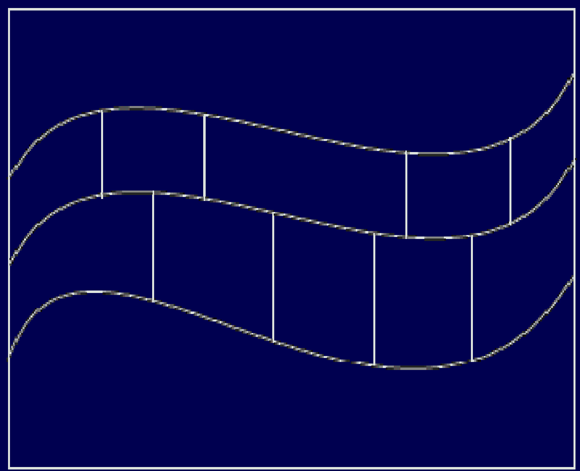
Strekalovskiy, Cremers, ICCV 2011



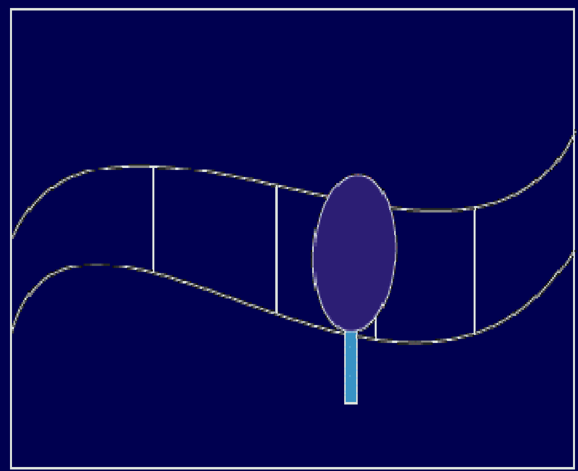
Five regions layout (Liu et al. [10])



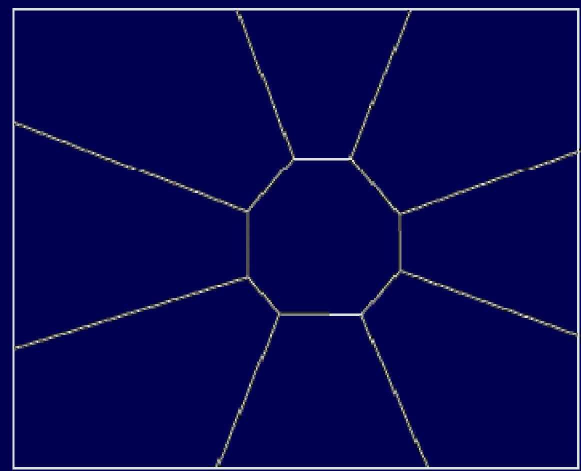
Tiered layout (Felzenszwalb et al. [4])



Four and more tiers

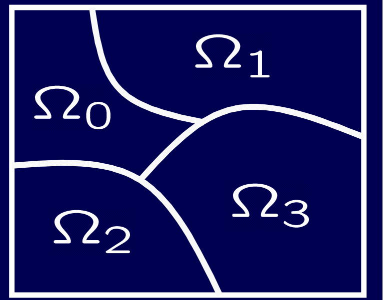


Floating objects



Convex shape prior

Strekalovskiy, Cremers, ICCV 2011



Reminder: With $v_i = 1_{\Omega_i}$, the segmentation problem is:

$$\min_{v \in \mathcal{B}} \frac{1}{2} \sum_i \int_\Omega |Dv_i| + \int_\Omega v_i \, f_i \, dx = \min_{v \in \mathcal{B}} \sup_{p \in \mathcal{K}} \sum_i \int_\Omega v_i \operatorname{div} p_i \, dx + \int_\Omega v_i \, f_i \, dx$$

where $\mathcal{K} = \left\{ p = (p_1, \ldots, p_n)^\top \in \mathbb{R}^{n \times m} \; : \; \left| p_i - p_j \right| \leq 1, \; \forall i < j \right\}$

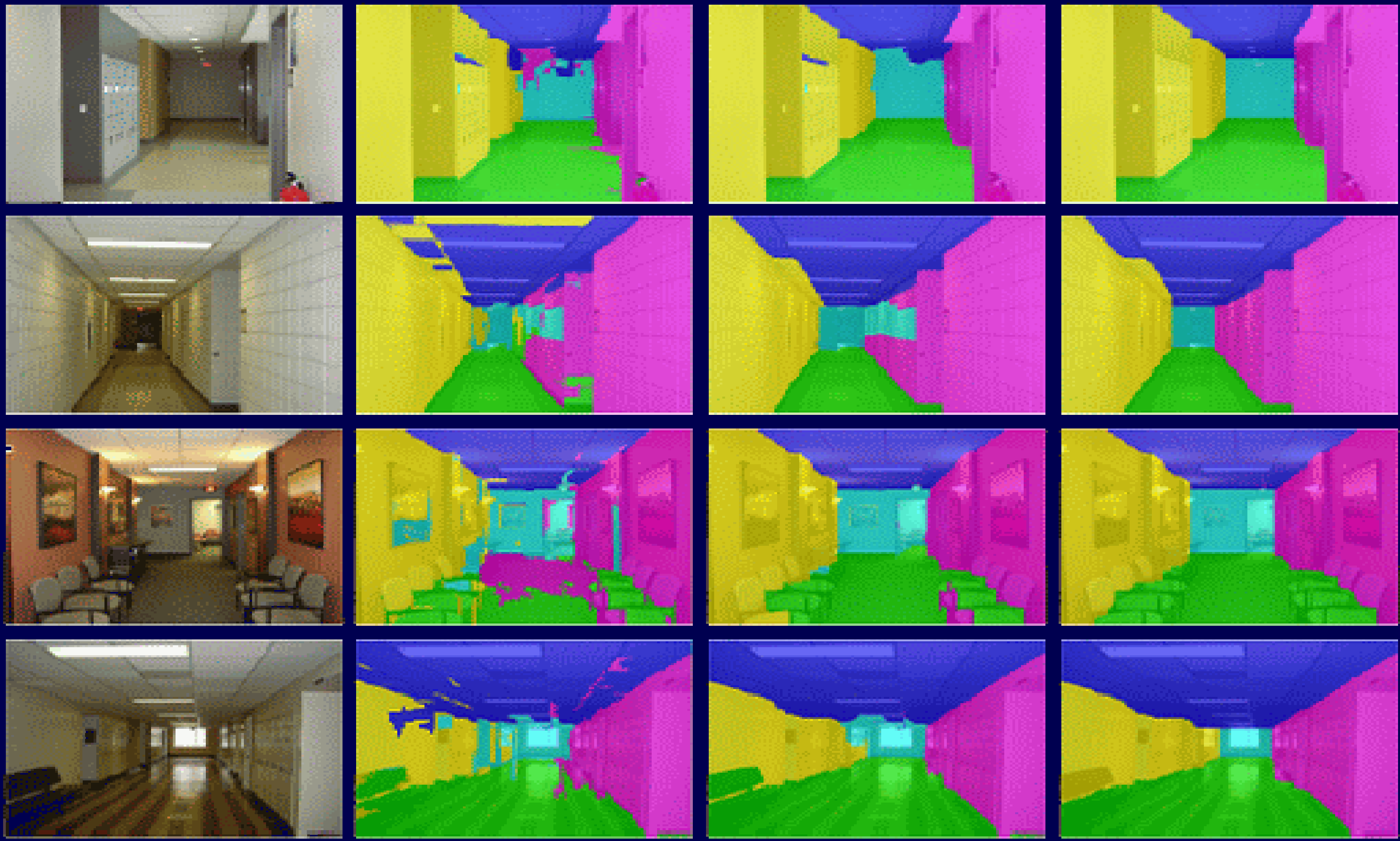
Consider instead the more general convex set:

$$\mathcal{K}_d = \left\{ p = \in \mathbb{R}^{n \times m} \; : \; \langle p_i - p_j, \nu \rangle \leq d(i, j, \nu), \; \forall i < j, \nu \in \mathbb{S}^{m-1} \right\}$$

Penalize transitions depending on label values i, j and orientation ν.

Strekalovskiy, Cremers, ICCV 2011

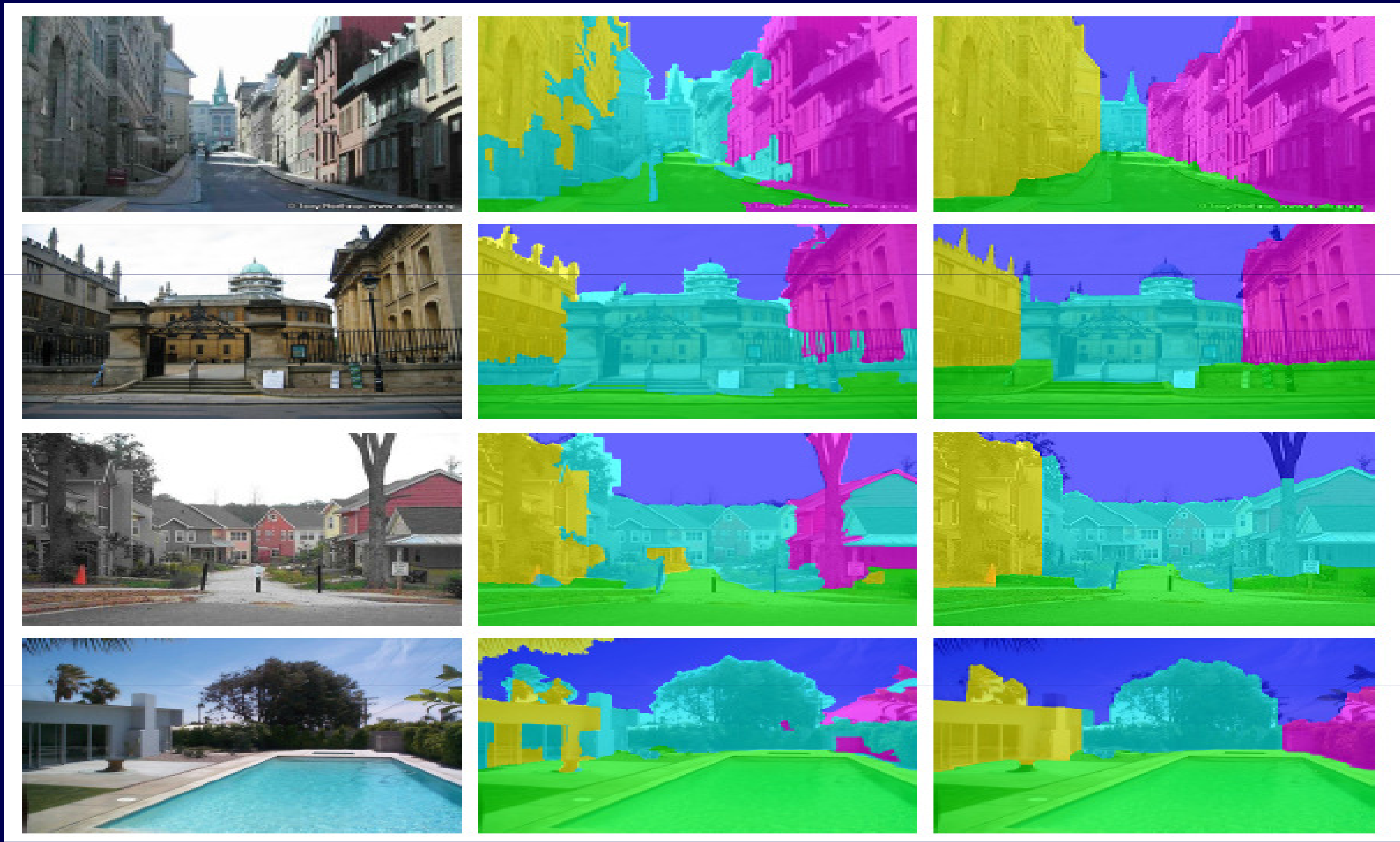
General Ordering Constraints



Input | Data term | Potts | Ordering

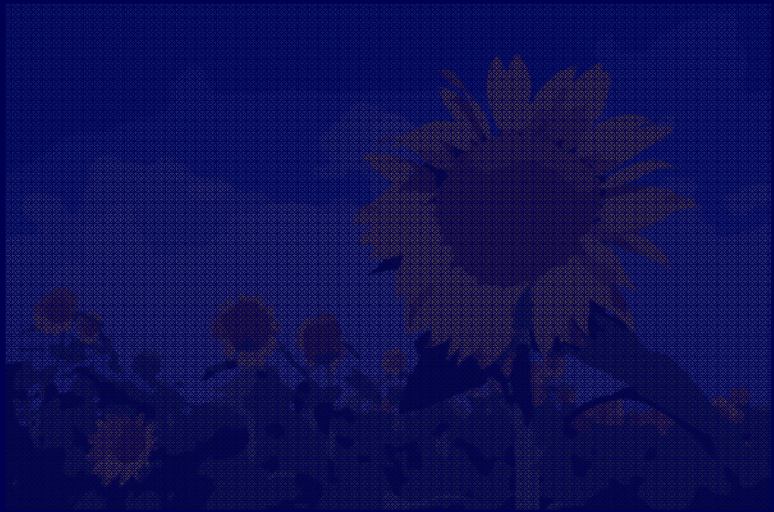
Strekalovskiy, Cremers, ICCV 2011

General Ordering Constraints

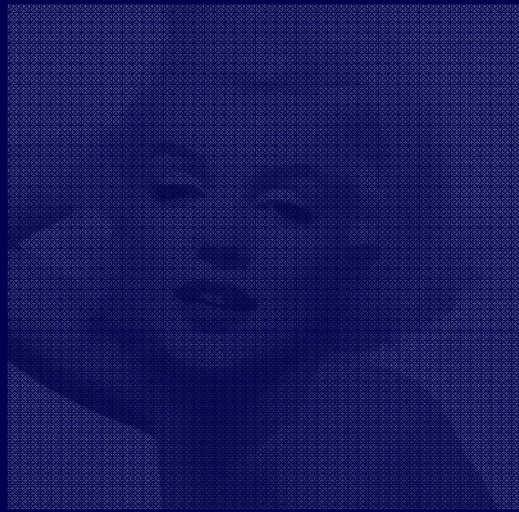


Input Potts Ordering

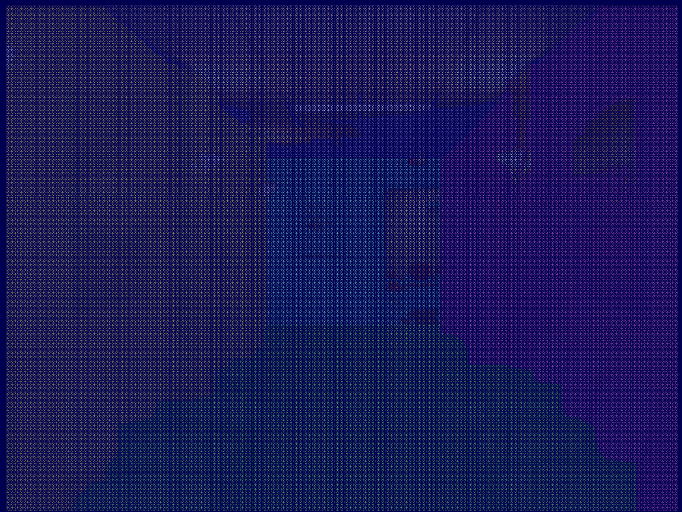
Strekalovskiy, Cremers, ICCV 2011



Convex multilabel optimization



Piecewise smooth approximation

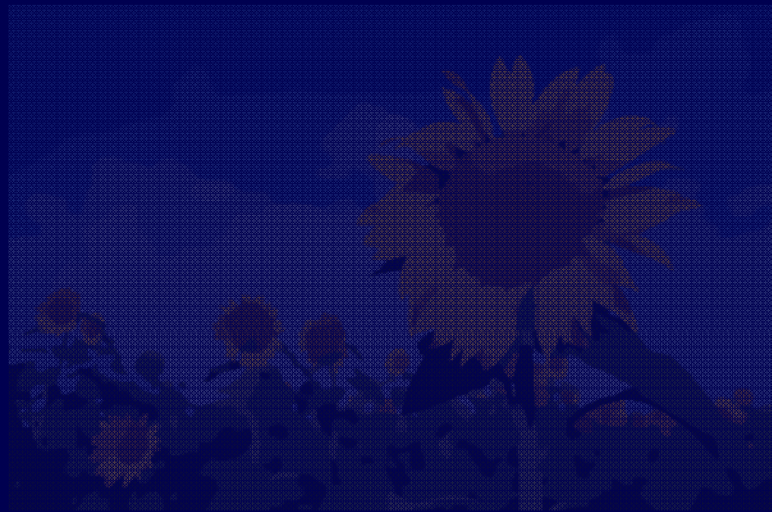


Convex ordering constraints

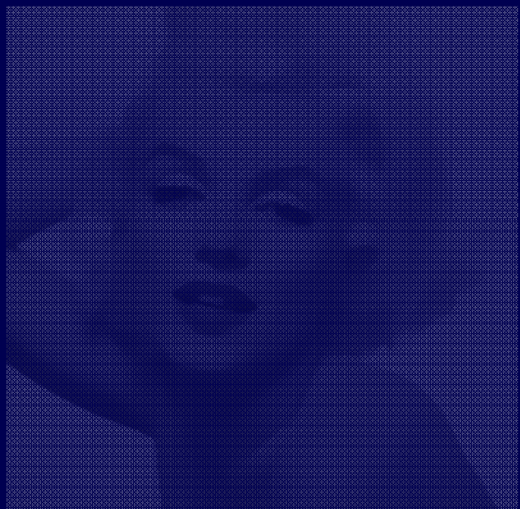
Convex optical flow

Overview

Convex multilabel optimization

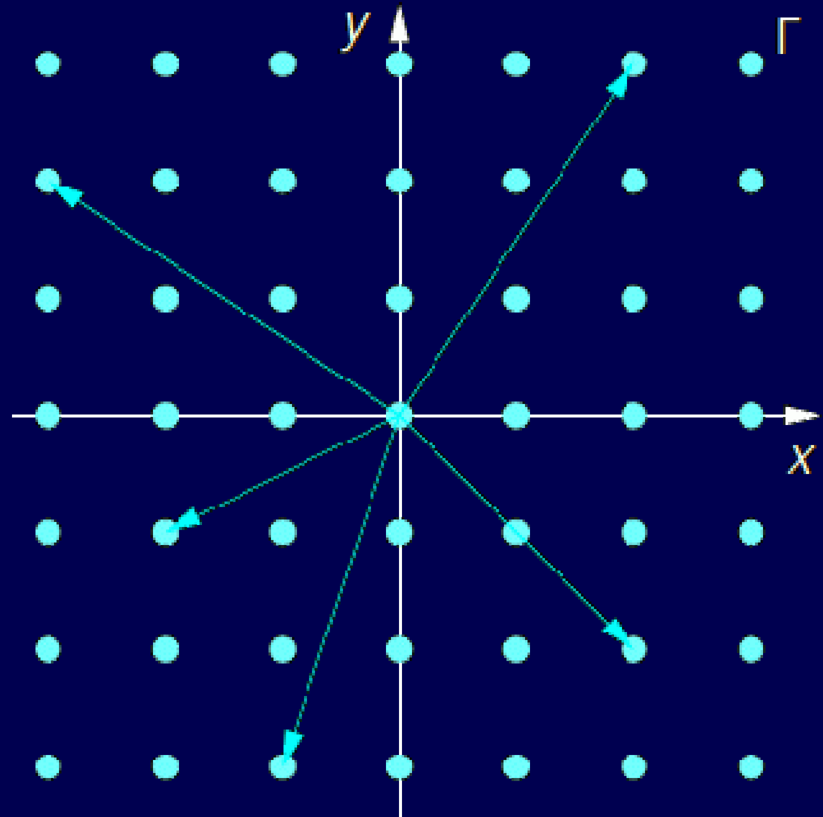


Convex multi-region segmentation



Piecewise smooth approximation

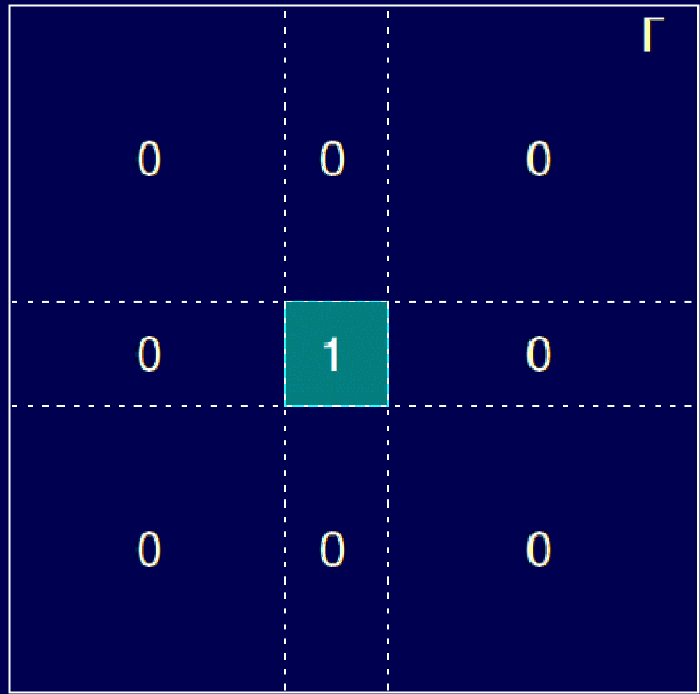
Convex optical flow



Optical flow: $$\min_{u:\Omega \to \Gamma \subset \mathbb{R}^d} \int_\Omega \left| I_1(x) - I_2(x+u) \right| dx + J(u)$$

Challenge: Thousands of labels cannot be handled in previous relaxations.

Goldluecke, Cremers ECCV '10, Strekalovskiy et al. ICCV '11



Optical flow: $$\min_{u:\Omega \to \Gamma \subset \mathbb{R}^d} \int_\Omega \left| I_1(x) - I_2(x+u) \right| dx + J(u)$$

Challenge: Thousands of labels cannot be handled in previous relaxations.

Goldluecke, Cremers ECCV '10, Strekalovskiy et al. ICCV '11

Convex Optical Flow

$$\min_{u:\Omega\to\Gamma} E_{data}(u) + E_{reg}(u) = \min_{u:\Omega\to\Gamma} \int_\Omega \rho(x,u)\,dx + \sum_{i=1}^{d} J(u_i)$$

Introduce: $v_i(x,\gamma_i) := \delta(u_i(x) - \gamma_i) \quad \forall\, i \in \{1,..,d\},\ \gamma_i \in \Lambda_i$

$$\min_{v_1,..,v_d} \int_{\Omega\times\Gamma} \rho(x,\gamma) \prod_{i=1}^{d} v_i(x,\gamma_i)\,dx\,d\gamma = \min_{v_1,..,v_d} \sup_{q\in Q} \left\{ \sum_{i=1}^{d} \int_{\Omega\times\Lambda_i} q_i\, v_i\, dx\, d\gamma_i \right\}$$

with: $Q = \left\{ (q_i : \Omega\times\Lambda_i \to \mathbb{R})_{i=1..d} \;\middle|\; \sum_{i=1}^{d} q_i(x,\gamma_i) \le \rho(x,\gamma)\ \forall x,\gamma \right\}$

$$\min E_{reg} = \min \sum_{i=1}^{d} \sup_{(p,b)\in C_i} \left\{ -\int_{\Omega\times\Lambda_i} \left(b + \operatorname{div} p\right) v_i(x,\gamma_i)\,dx\,d\gamma_i \right\}$$

Goldluecke, Cremers ECCV '10, Strekalovskiy et al. ICCV '11

Convex Optical Flow

Experimental optimality bounds ~ 3% - 5%

Goldluecke, Cremers ECCV '10, Strekalovskiy et al. ICCV '11

Convex Optical Flow

# of Pixels $P = P_x \times P_y$	# Labels $N_1 \times N_2$	Memory [Mb]		Run time [s]	
		Previous	Proposed (g/p)	Previous	Proposed (g/p)
320×240	8×8	112	112 / 102	196	26 / 140
320×240	16×16	450	337 / 168	*	80 / 488
320×240	32×32	1800	1124 / 330	*	215 / 1953
320×240	50×50	4394	2548 / 504	*	950 / 5188
320×240	64×64	7200	4050 / 657	-	1100 / 8090
640×480	8×8	448	521 / 413	789	102 / 560
640×480	16×16	1800	1351 / 676	*	295 / 1945
640×480	32×32	7200	4502 / 1327	-	1290 / 7795
640×480	50×50	17578	10197 / 2017	-	- / 32887
640×480	64×64	28800	16202 / 2627	-	- / 48583

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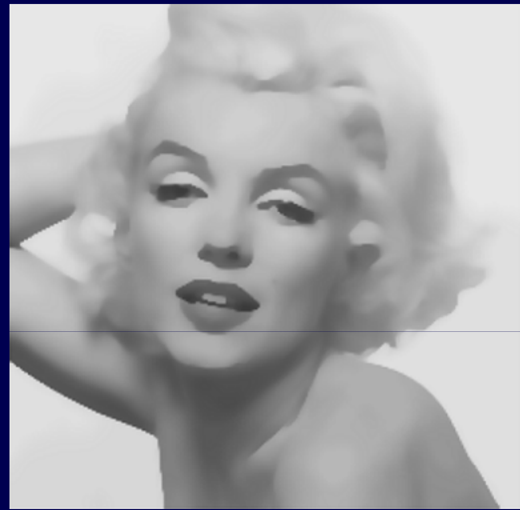
Conclusion

Convex multilabel optimization



Piecewise smooth approximation

Convex ordering constraints

Convex optical flow