Chapter 2

Continuous Optimization in Computer Vision

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Overview

Energy minimization methods

- Typical energies consist of a regularization term and a data term

$$\min_u \{E(u) = \mathcal{R}(u) + \mathcal{D}(u, f)\} \ ,$$

  where f is the input data and u is the unknown solution
- Energy functional is designed such that low-energy states reflect the physical properties of the problem
- Minimizer provides the best (in the sense of the model) solution to the problem

Energy minimization methods

- Discrete MRF setting: Images are represented as graphs $\mathcal{G}(\mathcal{V}, \mathcal{E})$, consisting of a node set $\mathcal{V}$, and an edge set $\mathcal{E}$. Each node $v \in \mathcal{V}$ can take a label from a discrete label set $\mathcal{U} \subset \mathbb{Z}$, i.e. $u(v) \in \mathcal{U}$
- Continuous Variational setting: Images are considered as continuous functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is the image domain.
- Link to statistics: In a Bayesian setting, the energy relates to the posterior probability via a Gibbs distribution

$$p(u|f) = \frac{1}{Z} exp(-E(u))$$

- Computing the minimizer of $E(u)$ is equivalent to MAP estimation on $p(u|f)$

The posterior distribution

- $p(u|f)$ provides a complete distribution.
- Which sample should we selected?
- Expectation: Compute sample, that minimizes the squared distance to the distribution

$$\bar{u} = \frac{1}{Z} \int_u u\, p(u|f)\, \mathrm{d}u$$

- Needs subtle algorithms to approximate the integral (MCMC)
- MAP: Computing that sample that maximizes the probability

$$u^* = \arg\max_u p(u|f) = \arg\min_u E(u)$$

- Leads to well-defined optimization algorithms

Example: Total variation based image restoration

Image model: $f = k * u + n$, blur kernel $k =$ 

Variational model: [Rudin, Osher, Fatemi '92]

$$\min_u \int_\Omega |Du| + \frac{\lambda}{2} \|k * u - f\|_2^2$$

(a) Degraded image f

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(a) Degraded image f

(b) Reconstructed image u

We need to solve large scale optimization problems

Optimization problems are unsolvable

Consider the following general mathematical optimization problem:

$$\begin{aligned} &\min f_0(x) \\ \text{s.t.} \quad &f_i(x) \leq 0 \,, \quad i = 1 \ldots m \\ &x \in S \,, \end{aligned}$$

where $f_0(x) \ldots f_m(x)$ are real-valued functions, $x = (x_1, \ldots x_n)^T \in \mathbb{R}^n$ is a n-dimensional real-valued vector, and S is a subset of $\mathbb{R}^n$

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- Naive: “Download a commercial package ...”

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How to solve this problem?

- Naive: “Download a commercial package ...”
- Reality: “Finding a solution is far from being trivial!”
- “Optimization problems are unsolvable”

[Nesterov ’04]

Complexity bounds for global optimization (1)

- Consider the problem class $\mathcal{C}_0$ [Nesterov '04]

$$\min_{x \in \mathcal{B}_n} f(x) \ ,$$

where $\mathcal{B}_n$ is the n-dimensional unit box defined by

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- What is the lower complexity bound to find an ε - approximate solution?

Complexity bounds for global optimization (2)

A remark on Lipschitz continuity: A function $f(x)$ is called Lipschitz continuous on $\mathcal{B}_n$ if

$$|f(x) - f(y)| \leq L\|x - y\|_\infty, \ \forall x, y \in \mathcal{B}_n \ ,$$

where the constant L is an upper bound to the maximum steepness of $f(x)$

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- **Theorem 1.1.2** [Nesterov '04]
  For $\frac{1}{2}L > \varepsilon > 0$, the complexity of a zero-order method to find an ε - approximate solution of the problem class $\mathcal{C}_0$ is at least

$$\left(\left\lfloor \frac{L}{2\varepsilon} \right\rfloor\right)^n$$

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- Comparison to NP-hard problems: Hard combinatorial problems need 2^n arithmetic operations (only)!

Convex versus non-convex

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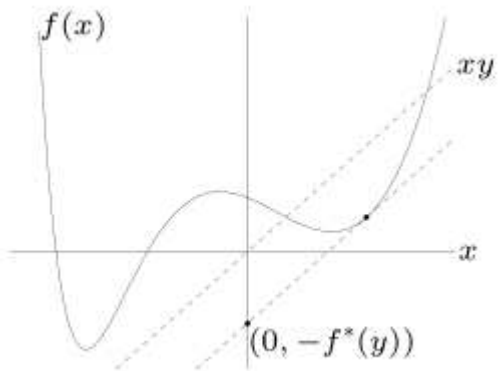
- Current research: Bridging the gap between convex and non-convex optimization
  - Convex approximations of non-convex models
  - New models
  - Algorithms
  - Bounds

Overview

The convex conjugate

- The convex conjugate $f^*(y)$ of a function $f(x)$ is defined through the Legendre-Fenchel transform

$$f^*(y) = \sup_{x \in \mathrm{dom}\ f} \langle x, y \rangle - f(x)$$



- $f^*(y)$ is a convex function (pointwise supremum over linear functions)

Examples

- $f(x) = |x|$:

$$f^*(y) = \sup_x \langle x, y \rangle - |x| = \begin{cases} 0 & \text{if} \quad |y| \leq 1 \\ \infty & \text{else} \end{cases}$$

- $f(x) = \frac{1}{2} x^T Q x$, Q, positive definite

$$f^*(y) = \sup_x \langle x, y \rangle - \frac{1}{2} x^T Q x = \frac{1}{2} y^T Q^{-1} y$$

The convex envelope

- The biconjugate function is defined by twice application of the Legendere-Fenchel transform

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$$f^{**}(x) = \sup_{y \in \mathrm{dom}\, f^*} \langle x, y \rangle - f^*(y)$$

- The biconjugate function $f^{**}(x)$ is the largest convex l.s.c. function below $f(x)$
- In case of uniqueness, $f(x)$ and $f^{**}(x)$ have the same minimum
- If $f(x)$ is a convex, l.s.c. function, $f^{**}(x) = f(x)$

$f(x)$ $\qquad$ $f^{**}(x)$

- Computing the biconjugate function allows to “convexify” a non-convex problem
- Unfortunately, computing $f^{**}(x)$ is not tractable for most problems

Approximate envelopes

- It is not possible to compute f^{**} in general (there are exceptions, though)
- What about approximate envelopes?
- Many problems can be decomposed as

$$f(x) = \sum_{i=1}^{n} f_i(x) \ ,$$

  where $f_i^{**}(x)$ is often much easier to compute than $f^{**}(x)$.
- It can be shown that

$$f(x) > f^{**}(x) = \left(\sum_i f_i \right)^{**} (x) > \left(\sum_i f_i^{**} \right) (x)$$

- This is exactly, what current research is about
- State-of-the art techniques (LP, QBPO, graph cuts, convex relaxations ...) exactly try to approximate the convex envelope in this way

Fenchel-Duality

- Consider the problem

$$\min_x f_1(x) - f_2(x)\,,$$

  where $f_{1,2}$ are real-valued functions
- Classical form of duality developed by W. Fenchel in the 40s

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- Introducing Lagrange multipliers y for the constraint $x_1 = x_2$ yields

$$\min_{x_1,x_2} \max_y f_1(x_1) - f_2(x_2) + \langle x_1 - x_2, y \rangle$$

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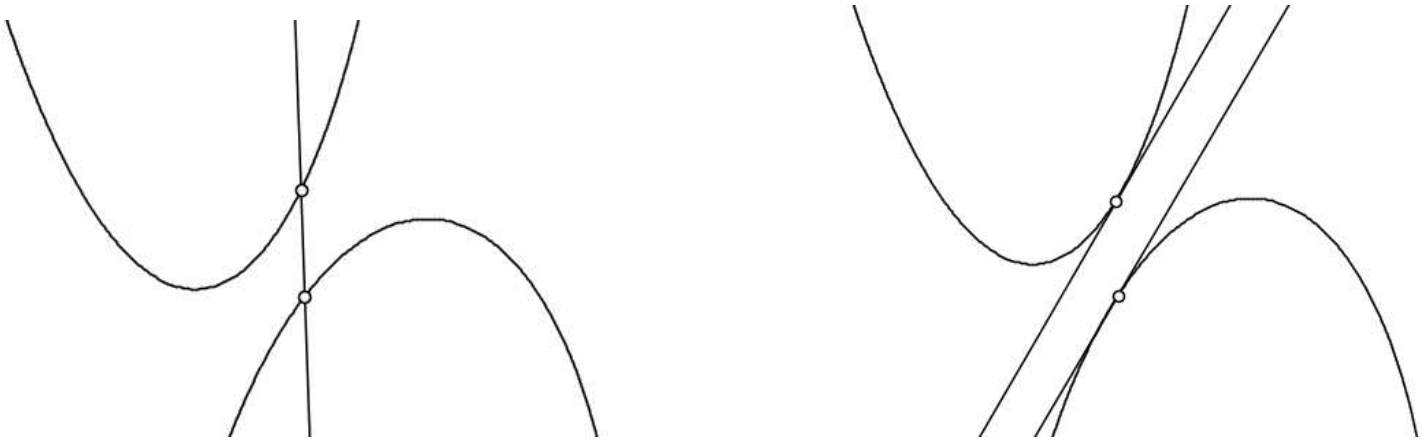
- We can recognize the definition of the convex conjugate functions, leads to the dual problem

$$\max_y f_2^*(y) - f_1^*(y)$$

Fenchel's duality theorem

If f_1 convex and f_2 concave, then

$$\min_x f_1(x) - f_2(x) = \max_y f_2^*(y) - f_1^*(y)$$



Interpretation: Minimizing the vertical distance between two points on the curves is equivalent to maximizing the distance between two parallel tangents on the curves

Overview

Convex optimization

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$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0 \,, \quad i = 1 \dots m \\ & Ax = b \end{aligned}$$

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- "The great watershed in optimization isn't between linearity and non-linearity, but convexity and non-convexity" [Rockafellar '93]

Black box convex optimization

Generic iterative algorithm for convex optimization:

1. Pick any initial vector $x^0 \in \mathbb{R}^n$, set $k = 0$
2. Compute search direction $d^k \in \mathbb{R}^n$
3. Choose step size τ^k such that $f(x^k + \tau^k d^k) < f(x^k)$
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Different methods to determine the search direction d^k

- steepest descend
- conjugate gradients
- Newton, quasi-Newton
- Working-horse for the lazy: Limited memory BFGS quasi-Newton method [Nocedal '80]

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Black box methods do not exploit the structure of the problem and hence are often less effective

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- ...
- Many problems can be formulated in these frameworks
- Fast methods available (simplex, interior point, ...)
- Sometimes ineffective for large-scale problems

Overview

A class of problems

Let us consider the following class of structured convex optimization problems

$$\min_{x \in X} F(Kx) + G(x) \ ,$$

- $K : X \to Y$ is a linear and continuous operator from a Hilbert space X to a Hilbert space Y.
- $F : Y \to \mathbb{R} \cup \{\infty\}$, $G : X \to \mathbb{R} \cup \{\infty\}$ are "simple" convex, proper, l.s.c. functions, and hence have an easy to compute prox operator:

$$\text{prox}_G(z) = (I + \partial G)^{-1}(z) = \arg\min_x \frac{\|x - z\|^2}{2} + G(x)$$

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- It turns out that many standard problems can be cast in this framework.
- There exists a vast literature of numerical algorithms to solve this class of problems

Examples

- Image restoration: The ROF model

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- General linear programming problems

$$\min_x \langle c, x\rangle \,, \text{ s.t. } \begin{cases} Ax &= b \\ x &\geq 0 \end{cases}$$

Primal, dual, primal-dual

The real power of convex optimization comes through duality

Recall the convex conjugate:

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$$\max_{y \in Y} -(F^*(y) + G^*(-K^* y)) \quad \text{(Dual)}$$

Allows to compute the duality gap, which is a measure of optimality

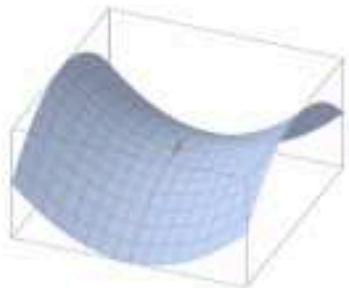
Optimality conditions

We focus on the primal-dual formulation:

$$\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle + G(x) - F^*(y)$$

We assume, there exists a saddle-point $(\hat{x}, \hat{y}) \in X \times Y$ which satisfies the Euler-Lagrange equations

$$\begin{cases} K\hat{x} - \partial F^*(\hat{y}) & \ni 0 \\ K^*\hat{y} + \partial G(\hat{x}) & \ni 0 \end{cases}$$



A note on Forward-backward splitting

- Consider the problem $\min_x f_1(x) + f_2(x)$, where $f_1(x)$ is a convex function and $f_2(x)$ is a convex function with L-Lipschitz continuous gradient ∇f_2, i.e.

$$\|\nabla f_2(x) - \nabla f_2(y)\| \leq L\|x - y\| \ , \forall x, y \in \mathrm{dom}(f_2)$$

- It can be shown that a minimizer can be characterized by the fixed point equation[Combettes, Pesquet '05]

$$x = (I + \tau \partial f_1)^{-1}(x - \tau \nabla f_2(x))$$

- This suggests the forward-backward iterations

$$x^{n+1} = \underbrace{(I + \tau \partial f_1)^{-1}}_{\text{backward step}} \underbrace{(x^n - \tau \nabla f_2(x^n))}_{\text{forward step}}$$

- **Note:** This is exactly the case for our saddle-point formulation

$$\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle + G(x) - F^*(y)$$

Arrow-Hurwicz algorithm

- Classical method for finding saddle-points [Arrow, Hurwicz, Uzawa '58]

$$\begin{cases} x^{n+1} = (I + \tau \partial G)^{-1}(x^n - \tau K^* y^n) \\ y^{n+1} = (I + \tau \partial F^*)^{-1}(y^n + \tau K x^{n+1}) \end{cases}$$

- Alternating forward-backward step in the primal variable and the dual variable
- Convergence under quite restrictive assumptions on τ
- For some problems very fast, e.g. ROF problem using adaptive time steps [Zhu, Chan, '08]

Proximal point algorithm

- Very general algorithm with well-studied properties [Martinet '70], [Rockafellar '76]
- Searches the zero of an operator, i.e. $0 \in T$
- General algorithm

$$p^{n+1} = (I + \tau^n T)^{-1} p^n$$

- In our setting:

$$p = \begin{pmatrix} x \\ y \end{pmatrix}, \quad T = \begin{pmatrix} \partial G & K^* \\ -K & \partial F^* \end{pmatrix}$$

- "Not implementable" since in most interesting cases, T is very hard to invert
- Many "implementable" algorithms can be shown to be a special case of the proximal point algorithm

Other interesting algorithms

- Extragradient methods [Korpelevich '76], [Popov '80]
  - Similar to Arrow-Hurwicz, but twice evaluation of the proximal operators
- Algorithms proposed by Yurii Nesterov
  - Many different algorithms, with guaranteed convergence rates, e.g. [Nesterov '03]
  - Recently, an increased interest in signal processing, computer vision and machine learning
- Douglas-Rachford splitting (DRS): [Mercier, Lions '79]
  - Can be applied if the problem can be decomposed into simpler problems
  - Special case of the proximal-point algorithm [Eckstein, Bertsekas '89]
  - Equivalent to an alternating minimization (ADMM) on the augmented Lagrangian formulation [Eckstein, Bertsekas '89]
  - Equivalent to the split-Bregman algorithm [Goldstein, Osher '09]

Other interesting algorithms

- Iterative shrinkage thresholding algorithm (ISTA) [Daubechies, Defrise, De Mol '04]
  - Developed for sparse recovery
  - Fast iterative shrinkage thresholding algorithm (FISTA) [Beck, Teboulle '09]
- Proximal splitting algorithms, e.g. [Combettes, Pesquet, '08]
  - Many different algorithms spezialized to interesting problems in image processing and sparse recovery

A simple primal-dual algorithm

Proposed in a series of papers: [Pock, Cremers, Bischof, Chambolle, '09], [Chambolle, Pock, '10], [Pock, Chambolle, '11]

- Initialization: Choose s.p.d. T, Σ, $\theta \in [0, 1]$, $(x^0, y^0) \in X \times Y$.
- Iterations ($n \geq 0$): Update x^n, y^n as follows:

$$\begin{cases} x^{n+1} &= (I + \mathrm{T}\partial G)^{-1}(x^n - \mathrm{T}K^* y^n) \\ y^{n+1} &= (I + \Sigma\partial F^*)^{-1}(y^n + \Sigma K(x^{n+1} + \theta(x^{n+1} - x^n))) \end{cases}$$

- Alternates gradient descend in x and gradient ascend in y
- Linear extrapolation of iterates of x in the y step
- T, Σ can be seen as preconditioning matrices

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- Alternates gradient descend in x and gradient ascend in y
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- T, Σ can be seen as preconditioning matrices

- Can be derived from a pre-conditioned ADMM algorithm
- Can be seen as a relaxed Arrow-Hurwicz scheme
- Can be seen as an approximate extragradient scheme

Relations to the proximal-point algorithm

- The iterations of PD can be written as the variational inequality [He,Yuan '10]

$$\left\langle \begin{pmatrix} x - x^{n+1} \\ y - y^{n+1} \end{pmatrix}, \begin{pmatrix} \partial G(x^{n+1}) + K^* y^{n+1} \\ \partial F^*(y^{n+1}) - K x^{n+1} \end{pmatrix} + M \begin{pmatrix} x^{n+1} - x^n \\ y^{n+1} - y^n \end{pmatrix} \right\rangle \geq 0 \,,$$

$$M = \begin{bmatrix} \mathrm{T}^{-1} & -K^* \\ -\theta K & \Sigma^{-1} \end{bmatrix}$$

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- Convergence is ensured if M is symmetric and positive definite
- This is the case if $\theta = 1$ and the assertion $\| \Sigma^{\frac{1}{2}} K \mathrm{T}^{\frac{1}{2}} \|^2 < 1$ is fulfilled.

A family of diagonal preconditioners

- It is important to choose T, Σ such that the prox-operators are still easy to compute
- Restrict the preconditioning matrices to diagonal matrices
- It turns out that: Let $K \in \mathbb{R}^{m \times n}$, $\mathrm{T} = \mathrm{diag}(\boldsymbol{\tau})$ and $\Sigma = \mathrm{diag}(\boldsymbol{\sigma})$ such that

$$\tau_j = \frac{1}{\sum_{i=1}^{m} |K_{i,j}|^{2-\alpha}}, \quad \sigma_i = \frac{1}{\sum_{j=1}^{n} |K_{i,j}|^{\alpha}}$$

then for any $\alpha \in [0, 2]$

$$\|\Sigma^{\frac{1}{2}} K \mathrm{T}^{\frac{1}{2}}\|^2 \leq 1 \,.$$

- Gives an automatic problem-dependent choice of the primal and dual steps
- Allows to apply the algorithm in a plug-and-play fashion
- For $\alpha = 0$, equivalent to the alternating step method [Eckstein, Bertsekas, '90]

Convergence rates

The algorithm gives different convergence rates on different problem classes [Chambolle, Pock, '10]

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- Coincides with so far best known rates of first-order methods

Parallel computing?

The algorithm basically computes matrix-vector products
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- One simple processor for each unknown
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Recent GPUs already go into this direction

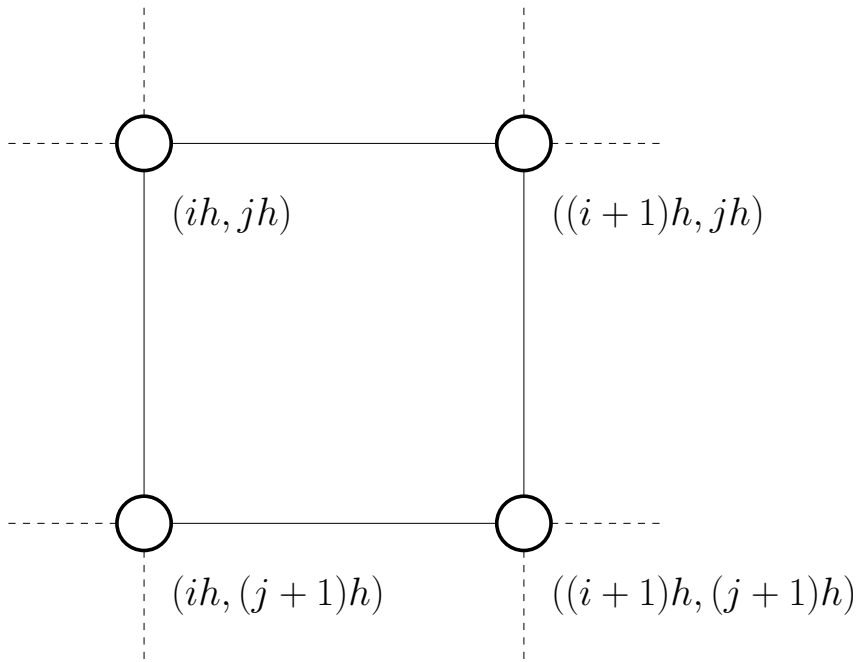
Overview

Discrete grid

We consider a regular Cartesian grid of size $M \times N$:

$$\Omega_h = \{(ih, jh) \,:\, 1 \leq i \leq M, 1 \leq j \leq N\}\,,$$

where h denotes the size of the spacing and (i, j) denote the indices of the discrete locations (ih, jh) in the image domain.



Discrete gradient

The gradient ∇u is a vector in the vector space $Y = X \times X$. For discretization of $\nabla : X \to Y$, we use standard finite differences with Neumann boundary conditions

$$(\nabla u)_{i,j} = \begin{pmatrix} (\nabla u)^1_{i,j} \\ (\nabla u)^2_{i,j} \end{pmatrix} ,$$

where

$$(\nabla u)^1_{i,j} = \begin{cases} \frac{u_{i+1,j} - u_{i,j}}{h} & \text{if } i < M \\ 0 & \text{if } i = M \end{cases}, \quad (\nabla u)^2_{i,j} = \begin{cases} \frac{u_{i,j+1} - u_{i,j}}{h} & \text{if } j < N \\ 0 & \text{if } j = N \end{cases} .$$

Furthermore we will also need the discrete divergence operator $\operatorname{div} p : Y \to X$, which is chosen to be adjoint to the discrete gradient operator. In particular, one has $-\operatorname{div} = \nabla^*$ which is defined through the identity

$$\langle \nabla u, p \rangle_Y = - \langle u, \operatorname{div} p \rangle_X .$$

Preconditioner: We easily find that $\mathrm{T} = \operatorname{diag}(\tau)$, where $\tau = h/4$ and $\Sigma = \operatorname{diag}(\sigma)$, where $\sigma = h/2$

The discrete ROF model

In the continuous setting, the ROF model is defined as

$$\min_u \int_\Omega |Du| + \frac{\lambda}{2} \|u - g\|_2^2$$

Using the discrete setting, we obtain

$$h^2 \min_{u \in X} \|\nabla u\|_1 + \frac{\lambda}{2} \|u - g\|_2^2 \,,$$

where the discrete total variation is given by

$$\|\nabla u\|_1 = \sum_{i,j} |(\nabla u)_{i,j}| \,, \quad |(\nabla u)_{i,j}| = \sqrt{((\nabla u)_{i,j}^1)^2 + ((\nabla u)_{i,j}^2)^2}$$

and

$$\|u - g\|_2^2 = \sum_{i,j} (u_{i,j} - g_{i,j})^2$$

The parameter λ is used to control the influence of regularization

The primal-dual ROF model

Casting the discrete ROF model in our framework, we obtain the following primal-dual ROF model

$$\min_{u \in X} \max_{p \in Y} -\langle u, \operatorname{div} p \rangle_X + \frac{\lambda}{2} \|u - g\|_2^2 - \delta_P(p) \,,$$

where the convex set P is given by

$$P = \{p \in Y : \|p\|_\infty \leq 1\} \,,$$

and $\|p\|_\infty$ denotes the discrete maximum norm defined as

$$\|p\|_\infty = \max_{i,j} |p_{i,j}| \,, \quad |p_{i,j}| = \sqrt{(p_{i,j}^1)^2 + (p_{i,j}^2)^2} \,.$$

Primal-dual algorithm to solve the ROF model

ROF-Primal-Dual

- Initialization: Choose $\tau = h/4$, $\sigma = h/2$, set $u^0 = \bar{u}^0 = f$, $p^0 = 0$, $\theta = 1$
- Iterations ($n \geq 0$)

$$\begin{cases} u^{n+1} &= (I + \tau \partial G)^{-1}(u^n - \tau(-\operatorname{div} p^n)) \\ p^{n+1} &= (I + \sigma \partial F^*)^{-1}(p^n + \sigma \nabla(2u^{n+1} - u^n)) \end{cases}$$

Computing the prox operators

Finally, we need to find explicit expressions for $(I + \sigma\partial F^*)^{-1}$ and $(I + \tau\partial G)^{-1}$. We have that $F^*(p) = \delta_P(p)$ and $G(u) = \frac{\lambda}{2}\|u - g\|_2^2$. Since F^* is the indicator function of a convex set, the resolvent operator reduces to pointwise Euclidean projectors onto L^2 balls

$$p = (I + \sigma\partial F^*)^{-1}(\tilde{p}) \Longleftrightarrow p_{i,j} = \frac{\tilde{p}_{i,j}}{\max(1, |\tilde{p}_{i,j}|)} \; .$$

The resolvent operator with respect to G poses simple pointwise quadratic problems. The solution is trivially given by

$$u = (I + \tau\partial G)^{-1}(\tilde{u}) \Longleftrightarrow u_{i,j} = \frac{\tilde{u}_{i,j} + \tau\lambda g_{i,j}}{1 + \tau\lambda} \; .$$

Practical implementation

- Matlab implementation → **Online-Demo**
  - Uses Matlab vector operations
  - Takes a few seconds for typical images
  - About 100 lines of code
  - Files can be downloaded from the Tutorial homepage
- Parallel implementation → **Online-Demo**
  - GPU implementation using Nvidia CUDA
  - Matlab fronted
  - About 1000 lines of code
  - Depending on the GPU 100-1000 times faster

(a) Clean (b) Noisy (c) Denoised

ROF-denoising on the GPU

Linear programming

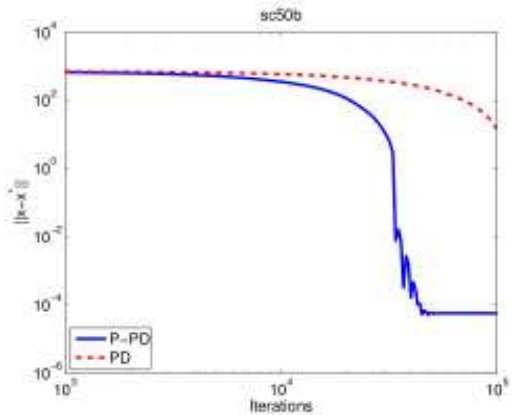
- Many applications of LP relaxations in computer vision and machine learning
- LP program in equality form

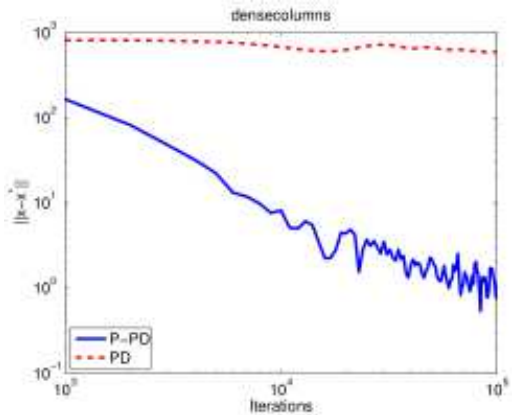
$$\min c^T x \quad \text{s.t.} \quad Ax = b\,,\; x \geq 0\,, \tag{1}$$

- Preconditioned primal-dual algorithm

$$\begin{cases} x^{k+1} = \text{proj}_{[0,\infty)}\left(x^k - \text{T}(A^T y^k + c)\right) \\ y^{k+1} = y^k + \Sigma(A(2x^{k+1} - x^k) - b)\,, \end{cases} \tag{2}$$

- Comparison





Graph cuts

- Graph cuts are widely used in computer vision [3]
- Can be written as a weighted total variation energy

$$\min_u \|D_w u\|_{\ell_1} + \langle u, w^u \rangle \ , \ \text{s.t. } 0 \leq u \leq 1 \ , \tag{3}$$

- Preconditioned primal-dual algorithm

$$\begin{cases} u^{k+1} = \text{proj}_{[0,1]} \left(u^k + \text{T}(D_w^T y^k - w^u) \right) \\ y^{k+1} = \text{proj}_{[-1,1]} \left(y^k + \Sigma(D_w(2u^{k+1} - u^k)) \right) \ , \end{cases} \tag{4}$$

- Comparison

MAXFLOW	PD	P-PD	P-PD-GPU
0.160s	15.75s	8.56s	0.045s

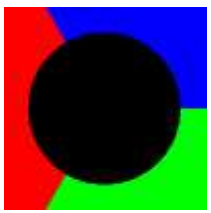
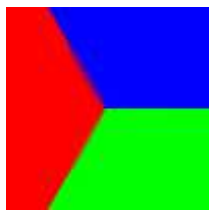
Continuous Potts model

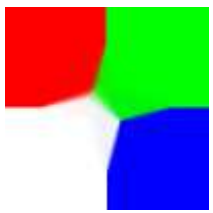
- The Continuous Potts model [4] with k phases can be written as a convex problem

$$\min_{u \in S} \max_{q \in B} \sum_{l=1}^{k} \langle Du_l, q_l \rangle + \langle u_l, f_l \rangle \quad , \tag{5}$$

where S is the simplex constraint and B is an interection of non-local ℓ_2 balls.

- Comparison



	PD	P-PD	Speedup
Synthetic (3 phases)	221.71s	75.65s	2.9
Synthetic (4 phases)	1392.02s	538.83s	2.6
Natural (8 phases)	592.85s	113.76s	5.2

Summary and open questions

- Energy minimization methods for computer vision
- Leads to large scale optimization problems
- Introduction into convex analysis and convex optimization
- Flexible primal-dual algorithm for a class of structured convex optimization problems

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- Can first-order methods go beyond the shown convergence rates?
- Combining first- and second-order methods?
- Discrete versus continuous optimization methods?