*Variational Methods in Computer Vision*
*ICCV Tutorial*, 06.11.2011

Chapter 4

# Convex Relaxation for Motion and Stereo

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# Overview



1. Motion estimation
2. Stereo estimation
3. Non-convex variational models

# Motion estimation

- Motion estimation (optical flow) is a central topic in computer vision,
- Computes a 2D vector field, describing the motion of pixel intensities

Applications:

- Tracking
- Video compression, video interpolation
- 3D reconstruction

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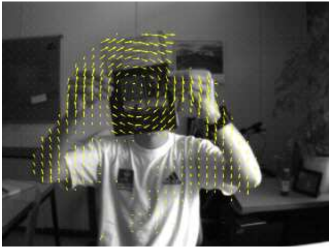

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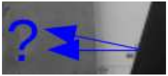
Motion estimation is still a very difficult problem

- Aperture problem

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- Illumination changes, shadows, ...
- Large motion of small objects, occlusions, ...

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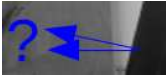


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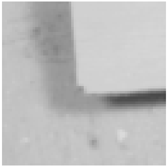

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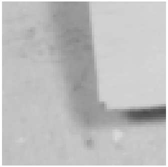

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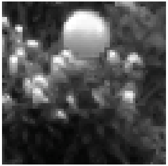

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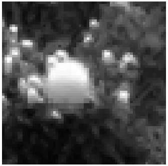
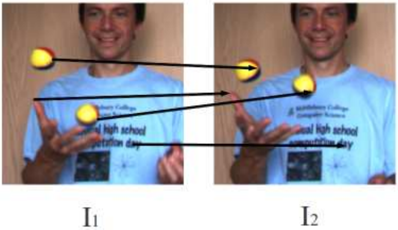


# The correspondence problem

- Find corresponding points in successive frames




- Brightness (color) constancy assumption

$$I_1(\mathbf{x}) - I_2(\mathbf{x} + \mathbf{u}(\mathbf{x})) \approx 0$$

- $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$ is the displacement vector
- Ambiguity: Many points with similar brightness (color)!
- Generalization: Constancy of image features (gradients, NCC, ...)

# Variational motion estimation

- Generic variational model for motion estimation

$$\min_{\mathbf{u}} \underbrace{\mathcal{R}(\mathbf{u})}_{\text{Regularization term}} + \underbrace{\int_\Omega |I_1(\mathbf{x}) - I_2(\mathbf{x} + \mathbf{u}(\mathbf{x}))|^p \, \mathrm{d}\mathbf{x}}_{\text{Data term}}$$

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  - Should favor physically meaningful flow fields
  - Popular convex regularizers: Quadratic, total variation, ...

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- Data term:
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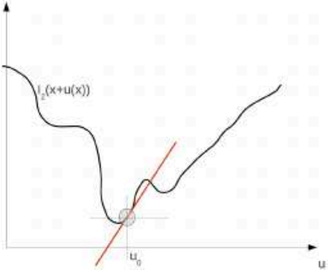
  - Different strategies to deal with the non-convexity of the data term
- Vast literature on motion estimation:
  - Window based optical flow: [Lucas, Kanade, 1981]
  - Variational optical flow: [Horn, Schunck, 1981]
  - Discontinuity preserving optical flow: [Shulman, Hervé '89]
  - Robust optical flow: [Black, Anadan, '93]
  - Highly accurate optical flow: [Brox et al. '04]
  - Real-time optical flow: [Bruhn et al.]
  - Primal-dual optimization on the GPU: [Zach, Pock, Bischof '07]

# Linearization of the image

- Perform a first order Taylor expansion of the function $I_2(\mathbf{x} + \mathbf{u}(\mathbf{x}))$ at $\mathbf{x} + \mathbf{u}_0(\mathbf{x})$
  [Horn, Schunck, 1981], [Lucas, Kanade, 1981]



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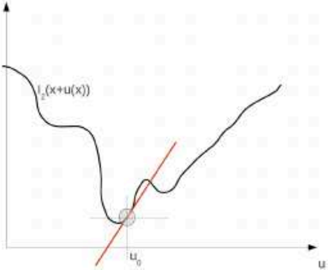


- $I_2(\mathbf{x} + \mathbf{u}(\mathbf{x})) \approx I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})) + \langle \nabla I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})), \mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}) \rangle$
- Only valid close to $\mathbf{u}_0$, i.e. $\|\mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x})\| \leq \varepsilon$

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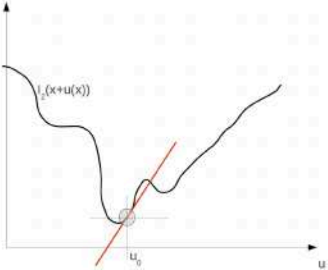


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- Only valid close to $\mathbf{u}_0$, i.e. $\|\mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x})\| \leq \varepsilon$
- Leads to the classical optical flow constraint:

$$\rho(\mathbf{u}) = I_1(\mathbf{x}) - I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})) - \langle \nabla I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})), \mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}) \rangle \approx 0$$

- Note: $\rho(\mathbf{u})$ is linear in $\mathbf{u}$ and hence $|\rho(\mathbf{u})|$ is convex!

# TV-$L^1$ motion estimation

- It turns out that total variation regularization in combination with a $L^1$ data term performs well
- Total variation allows for motion discontinuities
- $L^1$ data term allows for outliers in the data term (occlusions, noise, ...)

$$\min_{\|\mathbf{u}-\mathbf{u}_0\|\leq\varepsilon} \alpha \int_\Omega |D\mathbf{u}| + \|\rho(\mathbf{u})\|_1$$

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- Non-differentiable and hence difficult to solve
- Smoothing and fixed-point iteration: [Brox, Bruhn, Papenberg, Weickert ’04]
- Primal-dual optimization: [Chambolle, Pock, ’10]

$$\min_{\|\mathbf{u}-\mathbf{u}_0\| \leq \varepsilon} \max_{\|\mathbf{p}\|_\infty \leq \alpha} -\int_\Omega \mathbf{u} \operatorname{div} \mathbf{p} \, \mathrm{d}\mathbf{x} + \|\rho(\mathbf{u})\|_1$$

- Allows to compute the exact solution

# Second-order approximation of the data term

- Consider a more general non-convex data term of the form

$$\int_\Omega \phi(x, \mathbf{u}(x)) \, \mathrm{d}x$$

- Perform a second order Taylor expansion of the data term $\phi(x, \mathbf{u}(x))$ around $\mathbf{u}_0(x)$ [Werlberger, Pock, Bischof '10]

$$\phi(x, \mathbf{u}(x)) \approx \phi(x, \mathbf{u}_0(x)) + (\nabla\phi(x, \mathbf{u}_0(x)))^{\mathrm{T}} (\mathbf{u}(x) - \mathbf{u}_0(x)) + (\mathbf{u}(x) - \mathbf{u}_0(x))^{\mathrm{T}} \left(\nabla^2\phi(x, \mathbf{u}_0(x))\right) (\mathbf{u}(x) - \mathbf{u}_0(x)) \,,$$

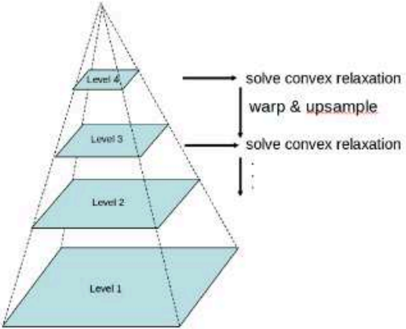
- To ensure convexity the Hessian $\nabla^2\phi(x, \mathbf{u}_0(x))$ has to be positive semidefinite
- We use the following diagonal approximation of the Hessian

$$\nabla^2\phi = \begin{bmatrix} (\phi_{xx}(x, u_0(x)))^+ & 0 \\ 0 & (\phi_{yy}(x, u_0(x)))^+ \end{bmatrix}$$

- Can be used with arbitrary data terms: SAD, NCC, ...
- Still only valid in a small neighborhood around $\mathbf{u}_0$
- Minimization using primal-dual schemes

# Large displacements

- How can we compute large displacements?
- Integrate the algorithm in a coarse-to fine / warping framework




- Similar to multigrid schemes, speeds up the minimization process
- Does not give any guarantees!

# Large displacement optical flow without warping

- Consider the following equivalent generic formulation [Steinbrücker, Pock, Cremers, '09]

$$\min_{\mathbf{u}} \mathcal{R}(\mathbf{u}) + \int_\Omega \phi(\mathbf{u}) \, \mathrm{d}\mathbf{x} \quad \Longleftrightarrow \quad \min_{\mathbf{u},\mathbf{v}} \mathcal{R}(\mathbf{u}) + \int_\Omega \phi(\mathbf{v}) \, \mathrm{d}\mathbf{x} \quad \text{s.t.} \quad \mathbf{u} = \mathbf{v}$$

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- Quadratic penality approach to obtain a unconstrained formulation

$$\min_{\mathbf{u},\mathbf{v}} \mathcal{R}(\mathbf{u}) + \frac{1}{2\theta}\|\mathbf{u} - \mathbf{v}\|_2^2 + \int_\Omega \phi(\mathbf{v})\, \mathrm{d}\mathbf{x}$$

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- Annealing-type scheme: Alternating minimization for a sequence of decreasing parameters $\theta_i$
- **Advantages:** No coarse-to-fine, no warping, arbitrary data terms
- **Disadvantage:** Results strongly depend on the sequence $\theta_i$

## Modified optical flow constraint

- Recall the optical flow constraint

$$\rho(\mathbf{u}) = I_1(\mathbf{x}) - I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})) - \langle \nabla I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})), \mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}) \rangle \approx 0$$

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- We can modify the constraint [Shulman, Hervé '89]

$$\delta(\mathbf{u}, v) = I_1(\mathbf{x}) - I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})) - \langle \nabla I_2(\mathbf{x} + \mathbf{u}_0(\mathbf{x})), \mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}) \rangle - v(\mathbf{x}) \approx 0$$

- $v(\mathbf{x})$ is a smooth function modeling illumination changes
- Note that $\delta(\mathbf{u}, v)$ is still linear in **u** and $v$!

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- $v(\mathbf{x})$ is a smooth function modeling illumination changes
- Note that $\delta(\mathbf{u}, v)$ is still linear in $\mathbf{u}$ and $v$!
- Additional regularization needed for $v(\mathbf{x})$

$$\min_{\|\mathbf{u}-\mathbf{u}_0\| \leq \varepsilon, v} \alpha \int_\Omega |D\mathbf{u}| + \beta \int_\Omega |Dv| + \|\delta(\mathbf{u}(\mathbf{x}), v(\mathbf{x}))\|_1$$

(a) Input (b) Ground truth (c) Estimated (d) Illumination

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# Stereo

- If $I_1$ and $I_2$ come from a stereo camera or a moving camera that browses a static scene, the displacement can be restricted to 1D problems on the epipolar lines, [Slesareva, Bruhn, Weickert '05]
- Each stereo pair can be normalized such that the displacement is only horizontally
- The depth $z$ can be computed from the displacement $u$ via

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- Optical flow constraint for stereo

$$\hat{\rho}(u) = I_1 - I_2(x+u_0(x,y),y) - \partial_x I_2(x+u_0(x,y),y)(u(x,y)-u_0(x,y)) \approx 0$$

- TV-$L^1$ based stereo

$$\min_{\|u-u_0\|\leq\varepsilon} \alpha \int_\Omega |Du| + \|\hat{\rho}(u(x,y))\|_1$$

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# Range estimation in a driving car (with Daimler AG)

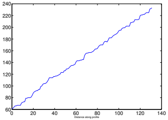
- Input images provided by a calibrated stereo rig

(a) Left image

(b) Right image

- Range image computed by the TV-$L^1$ based stereo algorithm

(a) Range image




(b) Profile of street

- Total variation regularization leads to the staircasing effect!

# Total generalized variation

- The total variation can be written (via the convex conjugate) as

$$\mathrm{TV}_\alpha(u) = \alpha \int_\Omega |Du| = \sup \left\{ \int_\Omega u \operatorname{div} v \, \mathrm{d}x \;\middle|\; v \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d), \|v\|_\infty \leq \alpha \right\},$$

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- In [Bredies, Kunisch, Pock, SIIMS'10], we proposed a generalization of the total variation to higher order smoothness.

$$\mathrm{TGV}_\alpha^k(u) = \sup \left\{ \int_\Omega u \,\mathrm{div}^k\, v \,\mathrm{d}x \;\middle|\; v \in \mathcal{C}_c^k(\Omega, \mathrm{Sym}^k(\mathbb{R}^d)), \|\mathrm{div}^l\, v\|_\infty \leq \alpha_l,\; l = 0, \ldots, k-1 \right\},$$

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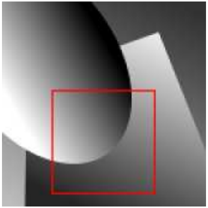
- For $k = 2$ it can be written as

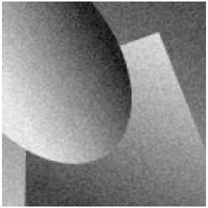
$$\mathrm{TGV}_\alpha^2(u) = \inf_{\mathbf{w}} \alpha_1 \int_\Omega |Du - \mathbf{w}| + \alpha_0 \int_\Omega |D\mathbf{w}|$$

- $\mathrm{TGV}^2$ can be used to reconstruct piecewise affine functions

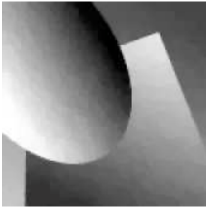
# Image restoration examples



(a) Clean image



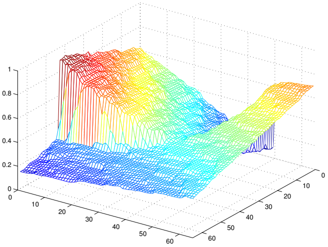
(b) Noisy image



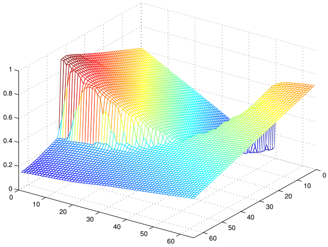
(c) TV

(d) $TGV^2$

# Image restoration examples




(a) TV




(b) $TGV^2$

# TGV **based stereo**

- Simply replace TV regularization by TGV regularization in the stereo model [Ranftl, Pock, Gehrig, Franke '11]

$$\min_{\|u-u_0\|\leq\varepsilon,\mathbf{w}} \alpha_1 \int_\Omega |Du - \mathbf{w}| + \alpha_0 \int_\Omega |D\mathbf{w}| + \|\hat{\rho}(u(x,y))\|_1$$

- Comparison on the stereo problem


(a) TV


(b) TGV$^2$

# Range estimation from a driving car

# Variational shape from focus (Stefan Heber)

- Joint work with Alicona Imaging, Graz
- Record an image sequence by varying the focus of an imaging system

- Variational model that computes a piecewise smooth surface which maximizes a certain sharpness measure $\sigma(x, u)$ on the surface

$$\min_u TGV(u) - \lambda \int_\Omega \sigma(x, u) \, \mathrm{d}\mathbf{x}$$

- Computation takes a few seconds on the GPU

# Overview

1. Motion estimation
2. Stereo estimation
3. Non-convex variational models

# Global solutions of non-convex variational models

- Consider the following non-convex energy-functional

$$\min_u \int_\Omega f(x, u(x), \nabla u(x)) \, dx$$

- We assume that $f(x, t, p)$ is convex in $p$ but non-convex in $t$

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- Example: TV-$L^1$ stereo

$$f(x, u(x), \nabla u(x)) = \alpha |\nabla u| + |I_1(x) - I_2(x + u(x))|$$

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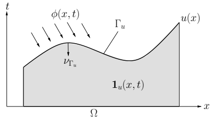
- What about the continuous setting?
- [Pock, Cremers, Bischof, Chambolle, SIIMS'10]

# The approach of Alberti, Bouchitte and Dal Maso

- The calibration method of [Alberti, Bouchitte, Dal Maso, '03], was orginally developed for the Mumford-Shah functional
- The basic idea is to consider the graph $\Gamma_u$ of $u$ instead of the function $u$
- Rewrite $E(u)$ by means of the flux of vector field $\phi$ through the graph $\Gamma_u$




- The characteristic function $\mathbf{1}_u$ of the subgraph of a function $u \in \mathcal{BV}(\Omega \times \mathbb{R}, [0,1])$ is defined as

$$\mathbf{1}_u(x,t) = \begin{cases} 1, & \text{if } t < u(x), \\ 0, & \text{else.} \end{cases}$$

- The normal $\nu_{\Gamma_u}$ of the interface $\Gamma_u$ is given by

$$\nu_{\Gamma_u} = \frac{(\nabla u, -1)}{\sqrt{|\nabla u|^2 + 1}}$$

# A lower bound

- Suppose, the maximum flux of a vector field $\phi = (\phi^x, \phi^t)$ through the graph provides a lower bound to $E(u)$

$$E(u) \geq \sup_{\phi \in \mathcal{K}} \int_{\Gamma_u} \phi \cdot \nu_{\Gamma_u} \, \mathrm{d}\mathcal{H}^2.$$

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- It turns out that equality holds for

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- The integral can be extended to $\Omega \times \mathbb{R}$

$$E(u) = \sup_{\phi \in K} \int_{\Omega \times \mathbb{R}} \phi \cdot D\mathbf{1}_u,$$

## Convex relaxation



- Relaxation of the binary function $\mathbf{1}_u : \Omega \to \{0, 1\}$ to functions $v : \Omega \to [0, 1]$

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  - The solution of the relaxed problem $v^*$ provides a solution to the binary problem $\mathbf{1}_u^*$ via the thresholding theorem

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  - The solution of the binary problem $\mathbf{1}_u^*$ provides a solution of the original non-convex problem $u^*$ via

$$u^* = \int_{\mathbb{R}} \mathbf{1}_u^*(x, t) \, \mathrm{d}t$$

# Illustrative example

- Consider the generic variation model

$$\min_u \int_\Omega h(\nabla u(x)) + \rho(u(x))\,\mathrm{d}x\,,$$

  where $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^+$ is a convex function
- Hence we have $f(x,t,p) = h(p(x)) + \rho(t(x))$
- Quadratic regularization: $h(p) = \frac{\|p\|_2^2}{2}$
- Recall the convex conjugate: $f^*(y) = \sup_x x \cdot y - f(x)$

$$\begin{aligned} f^*(x,t,\phi^x(x,t)) &= \sup_p \phi^x(x,t)\cdot p(x) - h(p(x)) - \rho(t(x)) \\ &= \frac{\|\phi^x(x,t)\|_2^2}{2} - \rho(t(x)) \end{aligned}$$

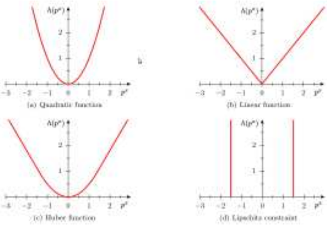
- The convex set $\mathcal{K}_q$ is then given by

$$\mathcal{K}_q = \left\{ \phi = (\phi^x, \phi^t) \;\middle|\; \phi^t(x,t) + \rho(t(x)) \geq \frac{\|\phi^x(x,t)\|_2^2}{2} \right\}$$

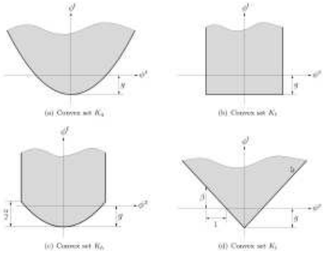
- Note: $\mathcal{K}_q$ is the union of simple pointwise constraints

# Visualization

- Potential functions $h(p)$



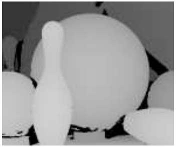

- Corresponding convex sets



# Stereo example

(a) Left input image

(b) Right input image



(c) True disparity

(d) Quadratic

(e) Total variation

(f) Huber

(g) Lipschitz

(h)

(i)

(j)

(k)

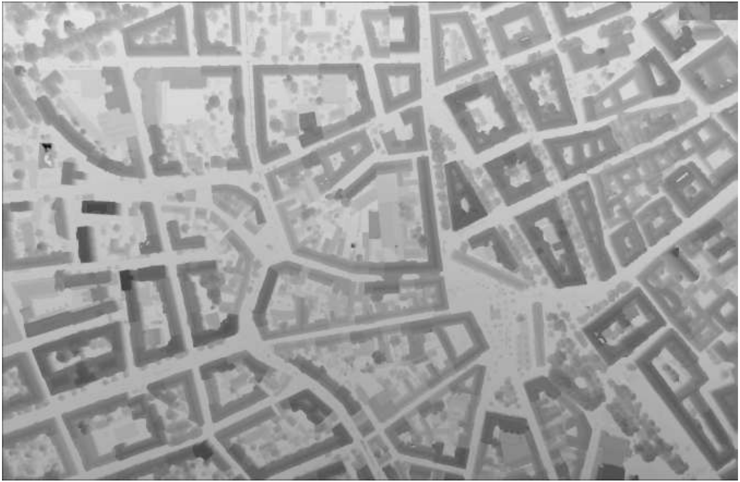
# Digital surface model of Graz

Input

# Digital surface model of Graz

Data term only

# Digital surface model of Graz



Convex variational approach

# Digital surface model of Graz

# Summary and open questions

- Introduced the problem of motion estimation in computer vision
- Motion estimation is still a challenging problem
- Highly non-convex data term leads to difficulties in the optimization
- A simple linearization approach works well in practice
- Can be used for stereo estimation
- TGV regularization avoids staircasing-artifacts
- Global solution of the stereo model using the calibration method

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- Global Solutions for Motion and Stereo?